PhD thesis topic

Quantum mechanics with non-self-adjoint operators

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In recent years there has been an enormous growth of interest in non-self-adjoint operators in quantum mechanics. The motivation lies not only in the possibility of unconventional representation of physical observables by non-self-adjoint operators [18, 1, 14], which has been overlooked for almost 70 years since the foundation of quantum mechanics, and in the recent availability of related experiments in optics [12, 17, 16], but also in the need of new ideas and methods to deal with the mathematical problems. The goal of the present research subject is to develop analytical tools needed in order to build a mathematically consistent and physically relevant quantum theory with non-self-adjoint operators. Possible topics of study are as follows:

Quasi-Hermitian and $\mathcal{PT}$-symmetric quantum mechanics

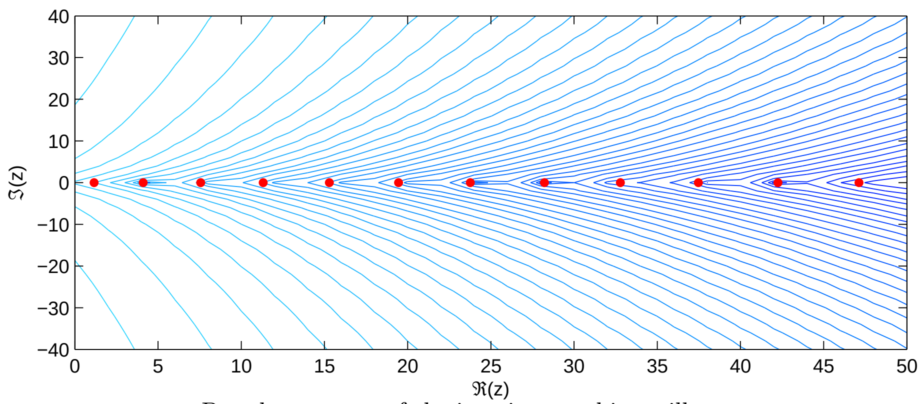
Nuclear physicists suggested in 1992 [18] an interesting *quasi-Hermitian* representation of observables in quantum mechanics, i.e., by non-self-adjoint operators H which satisfy the relation $H^* = \Theta\, H\, \Theta^{-1}$ with some positive, bounded and boundedly invertible operator Θ. A consistent quantum theory can be built for quasi-Hermitian observables by merely modifying the inner product $\langle\cdot,\cdot\rangle$ in an underlying Hilbert space to $\langle\cdot,\Theta\cdot\rangle$. In view of the similarity relation, the spectrum of a quasi-Hermitian operator is necessarily real.

The interest in quasi-Hermitian operators was renewed around the turn of the millennium when it was noticed [2] that there exists a large class of non-self-adjoint operators possessing *real spectra* and suggested that it is a consequence of an antilinear physical symmetry, so-called $\mathcal{PT}$-*symmetry*, i.e., $[H,\mathcal{PT}]=0$. The curious spectral property of $\mathcal{PT}$-symmetric operators was then attempted to be explained by the quasi-Hermiticity of the operators [13].

The task of the student will be to study the interplay between $\mathcal{PT}$-symmetry and quasi-Hermiticity on a rigorous level. In the situation when a given non-self-adjoint operator H does possess a metric operator Θ, the question consists in analysing the mathematical *structure* of Θ and the associated similarity transformation. Explicit results should be possible to obtain for specific models by developing the ideas of my recent paper [9] on Sturm-Liouville operators with non-self-adjoint boundary conditions.

Transitions from spectra to pseudospectra

In my recent paper [19], we proved a surprising lack of quasi-Hermiticity for a large class of Schrödinger operators with complex potentials by employing *semiclassical methods* and suggested the *pseudospectrum* [20, 5] as a more reliable quantity for non-self-adjoint operators in quantum mechanics. The student will apply the original ideas to other operators appearing in $\mathcal{PT}$-symmetric quantum mechanics and develop a numerical code for effective computations of pseudospectra.



Pseudospectrum of the imaginary cubic oscillator

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Traditionally self-adjoint techniques in the non-self-adjoint world

Because of the lack of variational methods, *quantitative bounds on the discrete spectrum* of non-self-adjoint operators constitute a highly non-trivial question. There exists a significant recent progress on the properties of discrete spectrum of non-self-adjoint operators based on *Lieb-Thirring inequalities* [8, 4, 11, 7, 6]. The task of the student will be to develop these methods for partial differential equations on *domains* with non-self-adjoint boundary conditions, in particular to $\mathcal{PT}$-symmetric waveguides [3].

The *Birman-Schwinger technique* seems to be particularly useful in this context; it was also used to establish the absence of discrete eigenvalues for Schrödinger operators with small complex potentials in higher dimensions [7]. In self-adjoint spectral theory, this type of result is associated with the existence of *Hardy inequalities*. At the same time, a related notion of *subcriticality* is meaningful as well for non-self-adjoint partial differential operators (with real coefficients) via the heat kernel [15]. It is thus relevant and important for applications to ask the question up to which extent the notions apply to general non-self-adjoint operators. Here the method of self-similarity variables to study the large-time behaviour of parabolic equations could be useful [10].

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