

PhD thesis topic

Operator theoretic approach to the theory of metamaterials

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Physical background and mathematical challenges

Metamaterials are modern artificial composite structures obtained by micro-structuring physical substances on a scale much less than the wavelength, which may exhibit some unusual features such as the negative refractive index leading to **optical antimatter** phenomena [10]. One of the theoretical approaches to the *cloaking effect* in metamaterials is given by Maxwell's equations in quasi-static regime with electrical permittivity and/or magnetic permeability assuming positive and negative signs in different parts of a spatial domain [9, 2]. Mathematically one therefore deals with a partial differential operator whose **principal symbol changes sign**.

Such problems **cannot be treated by standard methods**. Physicists overcome the obstacle by adding a small imaginary part to the permittivity and/or permeability and justify it by losses which always exist in real materials. Then the problem becomes mathematically treatable (by standard methods of sectorial forms), but the price one has to pay is that the operator in question is non-self-adjoint. The striking phenomenon of cloaking is revealed in the limit when the artificially added imaginary part is sent to zero. However, the **mathematical meaning of the limit is unclear**. Does one get a self-adjoint operator in the limit? If yes, what is the meaning of the cloaking as regards spectral properties of the self-adjoint operator? Does there exist a direct approach how to define and study the operator with sign-changing principal symbol?

The objective of this PhD research topic is to apply some mathematical techniques familiar in quantum mechanics to solve the challenging open problems in metamaterials.

Methodology

One obvious methodology of quantum mechanics to be adopted for metamaterials is the powerful technique of von Neumann theory of self-adjoint extensions of symmetric operators. The deep technical problem is that the operators arising in the theory of metamaterials are not bounded from below due to the sign-changing principal symbol and, moreover, essential spectra are expected to be present even for bounded domains. It demonstrates that a new, non-standard adaptation of the extension method should be developed.

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For one-dimensional systems this direction of research was already initiated by a research group of V. Kostrykin in Mainz, see [6] and [7]. As a breakthrough for higher-dimensional systems, in a collaboration with J. Behrndt [1], the responsible of the present project has recently succeeded in defining the cloaking operator on a rectangular geometry directly as a self-adjoint operator by means of a **refined extension theory** and established a peculiar property that the essential spectrum is not empty. Moreover, the very fresh preprint of other authors [3] indicates that the same method may work for more complicated geometries.

An alternative approach to follow is that of **indefinite quadratic forms** [5] with help of some known results from the theory of Navier-Stokes equations. Again, there exist some partial results by a student of V. Kostrykin [11]. See also [8] and [4] for other related works.

Our plan is to go further, develop and apply the aforementioned techniques to concrete geometries arising in metamaterials, and interpret the unusual spectral properties in terms of the cloaking phenomenon.

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