PhD thesis topic

Large-time behaviour of evolution systems

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11 April 2019

The objective of this PhD project is to cross-fertilise various approaches from several areas of mathematics to study the large-time behaviour of physical systems subject to non-trivial geometric and/or potential perturbations. We particularly intend to develop a systematic approach to determine the decay rate of the heat semigroup as regards a change of the underlying space or switch-on an electromagnetic field. In addition to classical diffusive models and stochastic processes, the study is motivated by the relevance of the heat semigroup in quantum mechanics [11].

A strong new motivation comes from mathematical models of nanostructures in which an electron transport is described by Schrödinger, Pauli or Dirac operators in non-trivial geometries and subject to external electromagnetic fields. In 2010 Krejčiřík and Zuazua [10] demonstrated that the Hardy inequality in twisted quantum waveguides has an influence on the large-time behaviour of solutions to the heat equation: a compactly supported deformation of twisting improves the polynomial decay rate with respect to the straight tube. Roughly saying, the heat flow (respectively, the Brownian motion) cools down (respectively, dies) more quickly in locally twisted tubes, a result which is perhaps expected from a physical intuition but very difficult to get mathematically. Similar results were obtained later by different authors and distinct methods in [5]. In addition to the geometric deformations, there are recent papers where a similar phenomenon has been observed in the case of local magnetic perturbations [8, 9, 2].

The study has left several open questions of physical importance. It is expected that the same gain in the polynomial decay rate will persist if the geometric or potential perturbations are not compactly supported but vanish quickly at infinity. However, superpolynomial decay rates are anticipated if the perturbation tend to zero but very slowly. Another open question is to show that there is a (subpolynomial) decay rate even if the total magnetic flux is an integer (in which case the Aharonov-Bohm effect is known to be unobservable) but the magnetic field is still present.

One direction of student's research will consist of confirming the above conjectures. Our idea is to apply the technique of self-similar variables [4, 3, 12], which transforms the heat equation into a non-autonomous parabolic problem with singularly time-scaled coefficients, and study asymptotic properties of the underlying scaled Schrödinger-type operator by spectral-theoretical methods. As an alternative approach, the idea is to use the approximative singular and regular perturbation methods [1, 13, 6, 7] to estimate the leading (and higher) order solutions to the studied problems and compare them with the rigorous approach discussed above. More generally, the objective of the project is to

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develop a systematic theory to study the large-time behaviour of solutions to the heat and Schrödinger equations by studying low-energy properties of the underlying elliptic operators.

This PhD topic is a part of the research project "Quantum mechanics with non-self-adjoint operators: transition from spectra to pseudospectra" supported by the Czech Science Foundation (No. 18-08835S, PI: Krejčiřík).

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