Some application of rational based number system

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(i) Delone perturbed lattice with rational scaling constants.

(ii) Review of rational based number system (with Ch.Frougny and J.Sakarovitch).

(iii) Distribution of fractional parts of $\langle x(p/q)^n \rangle$.

(iv) How to construct such a Delone set.

(v) Generalization to non-integral algebraic number base.

Delone set and Meyer set

A set $X \in \mathbb{R}$ is **relatively dense** if the distance between two adjacent points are bounded from above by a constant $R > 0$, and is **uniformly discrete** if the distance is bounded from below by a constant $r > 0$. X is called a **Delone set** if X is relatively dense and uniformly discrete.

If X is a Delone set and there is a finite set F with $X - X \subset X + F$, then X is called **Meyer set**.

$F = \emptyset$ then X forms a lattice. If $F = \{a\}$ then X is a translated lattice.

Lagarias [4] showed that X is Meyer iff X and $X - X$ is Delone. Later he also proved in [5] that if a Meyer set X satisfies $\beta X \subset X$ with $\beta > 1$, then β is a Pisot number or a Salem number.

X is a **perturbed lattice** if there is a bounded real function from $\mathbb{Z}$ to $\mathbb{R}$ such that $X = \{n + f(n) \mid n \in \mathbb{Z}\}$.

Theorem For coprime integers p, q with $p > q > 1$ there exists a Delone perturbed lattice X with $\frac{p}{q}X \subset X$.

$X - X$ is not contained in a finitely generated module over $\mathbb{Z}$. This implies that X has infinite local configurations, i.e., $(X - X) \cap B(0, R)$ is an infinite set for some R.

The statement seems not so easy. (I hope...)

You might imagine that this fact is connected to β-expansion. In fact, the set Y of $\frac{3}{2}$-integers is closed, discrete and relatively dense in $\mathbb{R}_{>0}$ and satisfies $\frac{3}{2}Y \subset Y$. However we do not know whether Y is uniformly discrete in $\mathbb{R}_{>0}$.

We do not know much on the beta expansion of 1 for $\beta = 3/2$. If $3/2 \in C_3$ in the sense of Bertrand-Blanchard [3] (i.e., run length of 0 in $d_\beta(1-0)$ is bounded), then Y is a Delone set in $\mathbb{R}_{>0}$. One can take $X = Y \cup -Y$.

Further $\mathbb{Z}_\beta$ must be a perturbed lattice. This approach seems difficult. We use rational based number system instead.

Rational based number system

Fix coprime integers p, q with $1 < q < p$ and $\mathcal{A} = \{0, 1, \ldots, p-1\}$, the set of alphabets. Then each positive integer has a following expression:

$$n = \sum_{i=0}^{m} \frac{a_i}{q} \left(\frac{p}{q}\right)^i \qquad a_i \in \mathcal{A} \tag{1}$$

i.e., we consider an analogy of decimal system and substitute 10^i by $\frac{1}{q}(\frac{p}{q})^i$ $(i = 0, 1, \ldots)$.

In other words, for a given positive integer n, start with

$n_0 = n$ and define inductively n_i and a_i through

$$n_{i+1} = \frac{qn_i - a_i}{p} \qquad a_i \in \mathcal{A} = \{0, 1, \ldots, p-1\}.$$

The sequence n_i is strictly decreasing n_i and reach 0 in finite steps. Let us try the case $p = 3$, $q = 2$.

$$\begin{aligned} 1 &= 2_{\ (3/2)} \\ 2 &= 21_{\ (3/2)} \end{aligned}$$

$$3 = 210_{(3/2)}$$

$$4 = 212_{(3/2)}$$

$$5 = 2101_{(3/2)}$$

$$6 = 2120_{(3/2)}$$

$$7 = 2122_{(3/2)}$$

$$8 = 21011_{(3/2)}$$

$$9 = 21200_{(3/2)}$$

Introduce a labeled tree $T(p/q)$ of p/q number system:

$$\mathbb{N} \ni z \xrightarrow{a} \frac{pz+a}{q} \in \mathbb{N}.$$

Edges are not drawn when $(pz+a)/q$ is not an integer. The set $L(p/q)$ of words which represent integers is given as all labels starting from the origin 0.

From this we can show several properties.

- $L(p/q)$ is prefix closed.
- Any word $\omega \in \mathcal{A}^*$ appears as subword of some element in $L(p/q)$.
- Let $\omega_1, \omega_2 \in \mathcal{A}^*$. If $\omega_1 \omega_2^n \in L(p/q)$ for each $n = 1, 2, \ldots$ then $\omega_1, \omega_2 \in 0^*$.

The last one implies a strong **aperiodicity** of $L(p/q)$. For short, **infinite repetition is not allowed** except 0^∞. By pumping lemma, $L(p/q)$ is not even context free.

Compactify this p/q number system by extending to the right.

$$\sum_{i=1}^{\infty} \frac{a_{-i}}{q} \left(\frac{p}{q}\right)^{-i} = \frac{a_{-1}}{p} + \frac{a_{-2}}{p}\frac{q}{p} + \frac{a_{-3}}{p}\left(\frac{q}{p}\right)^{2} + \dots.$$

where each prefix of this word in $L(p/q)$. This is written as by $.a_{-1}a_{-2}a_{-3}\dots$. Multiplying p/q power, every positive real number is expanded in this system.

In [2], we showed:

- Every positive real number has an **aperiodic** expansion in base p/q.

- The expression is unique but for countable exceptions.

- Exceptions are number theoretically characterized (Mahler's problem) when $p \geq 2q-1$.

The inequality $p \geq 2q-1$ is a technical condition and we expect it should be erased.

Theorem.(Frougny-Sakarovitch-A.) Let $k_c \in \{0, 1, \ldots, p-1\}$ be defined by $qk_c \equiv c \pmod{p}$ and assume that $p \geq 2q-1$. Then

$$\left\langle \frac{\xi}{q}\left(\frac{p}{q}\right)^n \right\rangle \in \bigcup_{c=0}^{q-1} \left[\frac{k_c}{p}, \frac{k_c+1}{p}\right)$$

holds for $n \geq n_0$ if and only if ξ has two different p/q expressions. Especially such numbers are countable and infinite.

If $p/q = 3/2$, then there exists ξ with

$$\left\langle \frac{\xi}{2}\left(\frac{3}{2}\right)^n \right\rangle \in [0, 1/3) \cup [2/3, 1). \quad (n = 1, 2, \ldots)$$

The assumption $p \geq 2q - 1$ guarantees that there are no positive number with three different expressions. We believe there is no number with three different expressions in all cases. A refinement of [2] was shown in [1]. It deals with the case when p is large.

Theorem

Let $p > q > 1$ with $p \geq 2q - 1$. Then a positive real number x has two p/q-representations if and only if there exist n_0 so that

$$\left\langle \frac{x}{q} \left(\frac{p}{q}\right)^n \right\rangle \in \bigcup_{0 \leq c \leq q-1} \left] \frac{k_c}{p}, \frac{k_c}{p} + \frac{q-1}{p(p-q)} \right[\tag{2}$$

holds for all $n \geq n_0$.

When p is larger, we can even prove:

Theorem

If $p > q^2$, the there exist ξ such that $\left\langle \frac{\xi}{q} \left(\frac{p}{q}\right)^n \right\rangle$ $n = 1, 2, \ldots$ always stays in a fixed Cantor space. Such ξ's correspond to double points of p/q number system.

Why is this expansion good ?

Let $x = .a_{-1}a_{-2}\dots$ and multiply some power of p/q:

$$\left(\frac{p}{q}\right)^m x = a_{-1}a_{-2}\dots a_{-m}.a_{-m-1}a_{-m-2}\cdots$$

As $a_{-1}\dots a_{-m} \in L(p/q)$, this integer part of p/q number system is in fact a rational integer. Therefore

$$\left\langle \left(\frac{p}{q}\right)^m x \right\rangle = \langle .a_{-m-1}a_{-m-2}\dots\rangle \quad !$$

How to construct the Delone set

Realize each node $z \in \mathbb{Z}$ of the graph $T(p/q)$ into $\mathbb{R}$, by the map

$$\phi : \mathbb{Z} \ni z = a_k a_{k-1} \dots a_0 \to z + \sum_{i=1}^{\infty} \frac{a_{-i}}{q} \left(\frac{p}{q}\right)^{-i}$$

where $a_{-1}a_{-2}\dots$ is the minimal path starting from z. Then it is obvious that $\frac{p}{q}\phi(\mathbb{Z}) \subset \phi(\mathbb{Z})$, $\phi(\mathbb{Z})$ is relatively dense in $\mathbb{R}$. It is also uniformly discrete if $p > 2q - 1$. The last condition assures that $\phi(z)$ and $\phi(z+1)$ are far.

In the case $p \leq 2q - 1$ then some technical discussion is required. Two points $\phi(z)$ and $\phi(z+1)$ could be close if the minimal paths starting from z and $z+1$ are adjacent for some large length K, i.e., z is minimally followed by $a_1 a_2 \dots a_K$ and $z+1$ is minimally followed by $(a_1 - p + q)(a_2 - p + q)\dots(a_K - p + q)$ for some large K.

In this case, we delete all the nodes $z+1$ having this property. One see that this removal does not destroy the relatively denseness and the resulting set is uniformly discrete.

Geometric construction.

One can construct our Delone set by an amusing algorithm. Let $p/q = 3/2$. Consider the initial set $Z = \bigcup_{k=0}^{\infty}[k+0, k+1/3) \cup [k+2/3, k+1)$ in $\mathbb{R}$. Then we define

$$Y = \cap_{i=0}^{\infty} \left(\frac{2}{3}\right)^i Z$$

We get a relatively dense set Y in $\mathbb{R}$ and $\frac{3}{2}Y \subset Y$. If two points are closer than a given small constant, then delete the larger one. What we get is the desired Delone set X.

Non-integral algebraic number systems

One can generalize this story to non-integral expanding algebraic numbers.

A parameter vector $(r_0, r_1, \ldots, r_{d-1}) \in \mathbb{R}^d$ gives a **Shift Radix System**, if the integer sequence generated by a recurrence:

$$0 \le r_0 z_n + r_1 z_{n+1} + \cdots + r_{d-1} z_{n+d-1} + z_{n+d} < 1$$

is always eventually falls into $(0, 0, \ldots, 0)$ for any initial vectors $(z_0, \ldots z_{d-1}) \in \mathbb{Z}^d$. The set of such parameters are denoted by $\mathcal{D}_d^0$.

Let $P(x) = p_d x^d + p_{d-1} x^{d-1} + \cdots + p_0 \in \mathbb{Z}[x]$ with $p_d > 0$ and put $\mathcal{A} = \{0, 1, \ldots, |p_0| - 1\}$. An analogy to the usual CNS we can prove that each element of $\mathcal{R} = \mathbb{Z}[x]/P(x)$ has a representative in $\mathcal{A}[x]$ if and only if $(p_d/p_0, p_{d-1}/p_0, \ldots, p_1/p_0) \in D_d^0$.

Note that $\mathcal{R}$ is not finitely generated over $\mathbb{Z}$ if $p_d > 1$. One can define a suitable $\mathbb{Z}$-submodule M of rank d in $\mathbb{R}$ so that

- Each element of M has a unique expansion.
- The language generated by M is right extensible.

Under this setting we can prove the analogy to p/q number system.

- $L(M)$ is prefix closed.

- Any word $\omega \in \mathcal{A}^*$ appears as subword of some element in $L(M)$.

- Let $\omega_1, \omega_2 \in \mathcal{A}^*$. If $\omega_1 \omega_2^n \in L(M)$ for each $n = 1, 2, \ldots$ then $\omega_1, \omega_2 \in 0^*$.

- Every element of $\mathbb{R}^d$ has an **aperiodic** expansion.

References

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