Asymptotic behavior of beta-integers

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- Let $\beta > 1$ and $x \geq 0$, any series

$$x = \sum_{i=-\infty}^{k} x_i \beta^i =: x_k x_{k-1} \dots x_0 \bullet x_{-1} \dots, \qquad x_i \in \mathbb{N},$$

is a β-*representation* of x

- β-*expansion* $\langle x \rangle_\beta$ of x = β-representation of x obtained by the greedy algorithm
- $\mathbb{Z}_\beta := \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k x_{k-1} \dots x_0 \bullet\}$

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for $\beta \in \mathbb{N}$: $\mathbb{Z}_\beta = \mathbb{Z}$

for $\beta \notin \mathbb{N}$:

- $\mathbb{Z}_\beta$ not periodic
- $\mathbb{Z}_\beta$ has no accumulation points

$$\mathbb{Z}_\beta^+ = \{b_n \mid n \in \mathbb{N}\}$$

  with $b_0 = 0$ and $b_{n+1} > b_n$
- distances in $\mathbb{Z}_\beta$ bounded by 1
- $\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$

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- β is a *Parry number* if the number of distances in $\mathbb{Z}_\beta$ finite
- Pisot numbers $\subset$ Parry numbers $\subset$ Perron numbers
- $\mathbb{Z}_\beta^+$ coded by an infinite word u_β:

$$\begin{array}{ccc} \Delta_0 & \to & 0 \\ \Delta_1 & \to & 1 \\ & \vdots & \\ \Delta_{m-1} & \to & m-1 \end{array}$$

- u_β = fixed point of one of 2 possible primitive substitutions (simple and non-simple Parry)

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If φ, $u_\beta = \varphi(u_\beta)$, of the form

$$
\begin{aligned}
\varphi(0) &= 0^{t_1}1 \\
\varphi(1) &= 0^{t_2}2 \\
&\vdots \\
\varphi(m-2) &= 0^{t_{m-1}}(m-1) \\
\varphi(m-1) &= 0^{t_m}
\end{aligned}
$$

with $t_i \in \mathbb{N}$ and $t_j t_{j+1} \cdots t_m \prec t_1 t_2 \cdots t_m$ for every $1 < j \leq m$ and $t_m \neq 0$, then

β is a *simple Parry number*

Parry polynomial of β:

$$p(x) = x^m - t_1 x^{m-1} - t_2 x^{m-2} - \cdots - t_m$$

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If β is a simple Parry number, then

$$c_\beta := \lim_{n\to+\infty} \frac{b_n}{n} = \frac{\beta-1}{\beta^m-1} p'(\beta).$$

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Let β be a simple Parry number. If β is a Pisot number such that its Parry and minimal polynomial coincide, then $(b_n - c_\beta n)_{n\in\mathbb{N}}$ is bounded.

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Let β be a simple Parry number. Then, $(b_n - c_\beta n)_{n\in\mathbb{N}}$ is bounded iff β is a Pisot number such that its Parry and minimal polynomial coincide.

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Theorem (Gazeau, Verger-Gaugry)

If β is a simple Parry quadratic unit, then

$$\mathbb{Z}_\beta^+ = \left\{ b_n = c_\beta n + \frac{1-\beta}{\beta^2+\beta} + \frac{\beta-1}{\beta} \left\{ \frac{n+1}{1+\beta} \right\}, \ n \in \mathbb{N} \right\},$$

where $c_\beta = \dfrac{\beta^2+1}{\beta^2+\beta}$.

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- Substitution matrix M associated with φ by $M_{ij}=|\varphi(i-1)|_{j-1}$, i.e.,

$$M = \begin{pmatrix} t_1 & 1 & 0 & \dots & 0 & 0 \\ t_2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{m-1} & 0 & 0 & \dots & 1 & 0 \\ t_m & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

  is primitive
- Perron-Frobenius:
  - M has an algebraically simple eigenvalue $\lambda > |\alpha|$ for any other eigenvalue α of M
  - only to λ corresponds a positive eigenvector
- Characteristic polynomial of M = Parry polynomial
- $(\beta^{m-1}, \dots, \beta, 1)$ = left eigenvector of M associated with β
- Queffélec: Vector of letter frequencies = left eigenvector $(\rho_0, \rho_1, \dots, \rho_{m-1})$ normalized by $\sum_{i=0}^{m-1} \rho_i = 1$

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- Fabre: $U_i := |\varphi^i(0)|$, then

$$n = \sum_{i=0}^{k} a_i U_i \quad \text{if} \quad \langle b_n \rangle_\beta = a_k \ldots a_1 a_0 \bullet$$

- for any $w \in \{0, 1, \ldots, m-1\}^*$

$$(|w|_0, |w|_1, \ldots, |w|_{m-1})M = (|\varphi(w)|_0, |\varphi(w)|_1, \ldots, |\varphi(w)|_{m-1})$$

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formula for $\frac{1}{c_\beta} b_n - n$?

- $\frac{1}{c_\beta} = \lim_{n\to\infty} \frac{n}{b_n} = \lim_{i\to\infty} \frac{U_i}{\beta^i}$
- Parry = minimal polynomial $\Rightarrow$ distinct roots $\Rightarrow$ M diagonalizable
- $$U_i = (1,0,\dots,0)P^{-1}\begin{pmatrix} \beta^i & 0 & \dots & 0 \\ 0 & \beta_2^i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_m^i \end{pmatrix} P(1,1,\dots,1)^T$$
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- $\frac{1}{c_\beta} = (1,0,\dots,0)P^{-1} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} P(1,1,\dots,1)^T$

Theorem

Let β be a simple Parry number. If β is a Pisot number such that its Parry and minimal polynomial coincide, then $(b_n - c_\beta n)_{n\in\mathbb{N}}$ is bounded.

formula for $\frac{1}{c_\beta} b_n - n$?

- $\frac{1}{c_\beta} = \lim_{n\to\infty} \frac{n}{b_n} = \lim_{i\to\infty} \frac{U_i}{\beta^i}$
- Parry = minimal polynomial $\Rightarrow$ distinct roots $\Rightarrow$ M diagonalizable
- $U_i = (1,0,\dots,0)P^{-1} \begin{pmatrix} \beta^i & 0 & \dots & 0 \\ 0 & \beta_2^i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_m^i \end{pmatrix} P(1,1,\dots,1)^T$
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Outline

- if $\langle b_n \rangle_\beta = a_k \dots a_1 a_0 \bullet$, then $\frac{1}{c_\beta} b_n - n = \sum_{i=0}^{k} a_i \left(\frac{\beta^i}{c_\beta} - U_i \right) =$

$$= (1,0,\dots,0) P^{-1} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & -z_2^{(n)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -z_m^{(n)} \end{pmatrix} P (1,1,\dots,1)^T,$$

  where $z_j^{(n)} = \sum_{i=0}^{k} a_i \beta_j^i$
- β Pisot $\wedge$ Parry polynomial $=$ minimal polynomial $\Rightarrow$ $\left(z_j^{(n)} \right)_{n \in \mathbb{N}}$ is bounded for $j \in \{2, \dots, m\}$

$$\frac{1}{c_\beta} b_n - n = \sum_{j=2}^{m} \frac{-z_j^{(n)}}{p'(\beta_j)} \frac{1-\beta_j^m}{1-\beta_j}$$

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$$\boxed{\frac{1}{c_\beta} b_n - n = \sum_{j=2}^{m} \frac{-z_j^{(n)}}{p'(\beta_j)} \frac{1 - \beta_j^m}{1 - \beta_j}}$$

Outline

No idea about asymptotic behavior of β-integers for non-Parry numbers β, i.e., when $\mathbb{Z}_\beta$ assumes infinitely many distinct distances between neighbors.

Suggestion: Start with the existence of $\lim_{n\to\infty} \frac{b_n}{n}$.

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