

Asymptotic behavior of beta-integers

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1 Definition of beta-integers

- 2 Properties of beta-integers
- 3 Parry numbers
- 4 Results on asymptotic behavior
- 5 Known results on asymptotic behavior

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1 Definition of beta-integers

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12 Open problem

• Let $\beta > 1$ and $x \ge 0$, any series

$$x = \sum_{i=-\infty}^{k} x_i \beta^i =: x_k x_{k-1} \dots x_0 \bullet x_{-1} \dots, \qquad x_i \in \mathbb{N},$$

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is a β -representation of x

 β-expansion (x)_β of x = β-representation of x obtained by the greedy algorithm

$$\mathbb{Z}_{\beta} := \{ x \in \mathbb{R} \mid \langle |x| \rangle_{\beta} = x_k x_{k-1} \dots x_0 \bullet \}$$

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for
$$\beta \in \mathbb{N}$$
: $\mathbb{Z}_{\beta} = \mathbb{Z}$

for $\beta \notin \mathbb{N}$:

- $\blacksquare \mathbb{Z}_{\beta}$ not periodic
- **Z** $_{\beta}$ has no accumulation points

$$\mathbb{Z}_{\beta}^{+} = \{ b_n \mid n \in \mathbb{N} \}$$

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with $b_0 = 0$ and $b_{n+1} > b_n$

- \blacksquare distances in \mathbb{Z}_β bounded by 1
- $\blacksquare \beta \mathbb{Z}_{\beta} \subset \mathbb{Z}_{\beta}$

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 \blacksquare distances in \mathbb{Z}_β bounded by 1

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Parry numbers

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$\blacksquare \beta$ is a *Parry number* if the number of distances in \mathbb{Z}_{β} finite

Pisot numbers ⊂ Parry numbers ⊂ Perron numbers
 Z⁺_β coded by an infinite word u_β:

$$\begin{array}{rrrr} \Delta_0 & \to & 0 \\ \Delta_1 & \to & 1 \\ & \vdots \\ \Delta_{m-1} & \to & m-1 \end{array}$$

 u_β = fixed point of one of 2 possible primitive substitutions (simple and non-simple Parry)

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- $\blacksquare \beta$ is a *Parry number* if the number of distances in \mathbb{Z}_{β} finite
- Pisot numbers ⊂ Parry numbers ⊂ Perron numbers
- **•** \mathbb{Z}^+_{β} coded by an infinite word u_{β} :



• u_{β} = fixed point of one of 2 possible primitive substitutions (simple and non-simple Parry)

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β is a Parry number if the number of distances in Z_β finite
Pisot numbers ⊂ Parry numbers ⊂ Perron numbers
Z⁺_β coded by an infinite word u_β:

$$egin{array}{cccc} \Delta_0 & o & 0 \ \Delta_1 & o & 1 \ & dots & & dots & & \ & dots & & \ & dots & & \ & & dots & & \ & & \ & & dots & & \ & \$$

 u_β = fixed point of one of 2 possible primitive substitutions (simple and non-simple Parry)

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Simple Parry numbers

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Simple Parry numbers

If φ , $u_{\beta} = \varphi(u_{\beta})$, of the form

with $t_i \in \mathbb{N}$ and $t_j t_{j+1} \cdots t_m \prec t_1 t_2 \cdots t_m$ for every $1 < j \le m$ and $t_m \ne 0$, then

 β is a simple Parry number

Parry polynomial of β :

$$p(x) = x^m - t_1 x^{m-1} - t_2 x^{m-2} - \dots - t_m$$

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Simple Parry numbers

If φ , $u_{\beta} = \varphi(u_{\beta})$, of the form

$$egin{array}{rcl} arphi(0) &=& 0^{t_1}1 \ arphi(1) &=& 0^{t_2}2 \ && dots \ arphi(m-2) &=& 0^{t_{m-1}}(m-1) \ arphi(m-1) &=& 0^{t_m} \end{array}$$

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Theorem

If β is a simple Parry number, then

$$c_{eta}:=\lim_{n
ightarrow+\infty}rac{b_n}{n}=rac{eta-1}{eta^m-1}p'(eta).$$

Theorem

Let β be a simple Parry number. If β is a Pisot number such that its Parry and minimal polynomial coincide, then $(b_n - c_\beta n)_{n \in \mathbb{N}}$ is bounded.

Conjecture

Let β be a simple Parry number. Then, $(b_n - c_\beta n)_{n \in \mathbb{N}}$ is bounded iff β is a Pisot number such that its Parry and minimal polynomial coincide.

Theorem

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- 8 Sketch of the proof 1st Theorem
- 9 Sketch of the proof tools
- 10 Sketch of the proof 2nd Theorem
- Sketch of the proof 2nd Theorem

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12 Open problem

Known results on asymptotic behavior

Theorem (Gazeau, Verger-Gaugry)

If β is a simple Parry quadratic unit, then

$$\mathbb{Z}_{\beta}^{+} = \left\{ b_{n} = c_{\beta}n + \frac{1-\beta}{\beta^{2}+\beta} + \frac{\beta-1}{\beta} \left\{ \frac{n+1}{1+\beta} \right\}, \ n \in \mathbb{N} \right\},$$
where $c_{\beta} = \frac{\beta^{2}+1}{\beta^{2}+\beta}.$

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Sketch of the proof – tools

Outline

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Sketch of the proof – tools

Substitution matrix M associated with φ by $M_{ij} = |\varphi(i-1)|_{j-1}$, i.e.,

$$M = \begin{pmatrix} t_1 & 1 & 0 & \dots & 0 & 0 \\ t_2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{m-1} & 0 & 0 & \dots & 1 & 0 \\ t_m & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

is primitive

Perron-Frobenius:

M has an algebraically simple eigenvalue λ > |α| for any other eigenvalue α of M

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- \blacksquare only to λ corresponds a positive eigenvector
- Characteristic polynomial of M = Parry polynomial
- $(eta^{m-1},\ldots,eta,1)=$ left eigenvector of M associated with eta
- Queffélec: Vector of letter frequencies = left eigenvector $(\rho_0, \rho_1, \dots, \rho_{m-1})$ normalized by $\sum_{i=0}^{m-1} \rho_i = 1$

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Sketch of the proof – 1st Theorem

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12 Open problem

Theorem

If β is a simple Parry number, then

$$c_{\beta} := \lim_{n \to +\infty} \frac{b_n}{n} = \frac{\beta - 1}{\beta^m - 1} p'(\beta).$$

• $b_n = |u|_0 \Delta_0 + |u|_1 \Delta_1 + \dots + |u|_{m-1} \Delta_{m-1}$, where u prefix of u_β of length n

- $\Box c_{\beta} = \rho_0 \Delta_0 + \rho_1 \Delta_1 + \dots + \rho_{m-1} \Delta_{m-1}$
- $(\rho_0, \rho_1, \dots, \rho_{m-1}) = \frac{1}{\sum_{i=0}^{m-1} \beta^i} (\beta^{m-1}, \dots, \beta, 1)$
- $\blacksquare \Delta_i \text{ known (Thurston)}$
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12 Open problem

Fabre: $U_i := |\varphi^i(0)|$, then

$$n = \sum_{i=0}^{k} a_i U_i$$
 if $\langle b_n \rangle_{\beta} = a_k \dots a_1 a_0 \bullet$

• for any $w \in \{0,1,\ldots,m-1\}^*$

 $(|w|_0, |w|_1, \dots, |w|_{m-1})M = (|\varphi(w)|_0, |\varphi(w)|_1, \dots, |\varphi(w)|_{m-1})$

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 $U_i = (1, 0, \dots, 0) M^i (1, 1, \dots, 1)^T$

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Sketch of the proof – 2nd Theorem

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12 Open problem

Theorem

Let β be a simple Parry number. If β is a Pisot number such that its Parry and minimal polynomial coincide, then $(b_n - c_\beta n)_{n \in \mathbb{N}}$ is bounded.

formula for $\frac{1}{c_{\beta}}b_n - n$?

$$\frac{1}{c_{\beta}} = \lim_{n \to \infty} \frac{n}{b_n} = \lim_{i \to \infty} \frac{U_i}{\beta^i}$$

■ Parry = minimal polynomial ⇒ distinct roots ⇒ M diagonalizable

$$U_{i} = (1, 0, ..., 0) P^{-1} \begin{pmatrix} \beta^{i} & 0 & ... & 0 \\ 0 & \beta_{2}^{i} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \beta_{m}^{i} \end{pmatrix} P(1, 1, ..., 1)^{T}$$

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Sketch of the proof – 2nd Theorem

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12 Open problem

Sketch of the proof – 2nd Theorem

• if
$$\langle b_n
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•, then $rac{1}{c_eta} b_n - n = \sum_{i=0}^k a_i \left(rac{eta^i}{c_eta} - U_i
ight) =$

$$= (1, 0, \dots, 0) P^{-1} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 - z_2^{(n)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -z_m^{(n)} \end{pmatrix} P(1, 1, \dots, 1)^T,$$

where
$$z_j^{(n)} = \sum_{i=0}^k a_i \beta_j^i$$

 β Pisot \wedge Parry polynomial = minimal polynomial $\Rightarrow (z_j^{(n)})_{n \in \mathbb{N}}$ is bounded for $j \in \{2, \ldots, m\}$

$$\frac{1}{c_{\beta}}b_{n} - n = \sum_{j=2}^{m} \frac{-z_{j}^{(n)}}{p'(\beta_{j})} \frac{1 - \beta_{j}^{m}}{1 - \beta_{j}}$$

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$$=(1,0,\ldots,0)P^{-1}\begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & -z_2^{(n)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -z_m^{(n)} \end{pmatrix}P(1,1,\ldots,1)^T,$$

where
$$z_j^{(n)} = \sum_{i=0}^k a_i \beta_j^i$$

 β Pisot \wedge Parry polynomial = minimal polynomial $\Rightarrow (z_j^{(n)})_{n \in \mathbb{N}}$ is bounded for $j \in \{2, \dots, m\}$

$$rac{1}{c_{eta}}b_n - n = \sum_{j=2}^m rac{-z_j^{(n)}}{p'(eta_j)} rac{1 - eta_j^m}{1 - eta_j}$$

Open problem

Outline

- 1 Definition of beta-integers
- 2 Properties of beta-integers
- 3 Parry numbers
- 4 Simple Parry numbers
- 5 Results on asymptotic behavior
- 6 Known results on asymptotic behavior
- 7 Sketch of the proof tools
- 8 Sketch of the proof 1st Theorem
- 9 Sketch of the proof tools
- 10 Sketch of the proof 2nd Theorem
- 11 Sketch of the proof 2nd Theorem

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12 Open problem

No idea about asymptotic behavior of β -integers for non-Parry numbers β , i.e., when \mathbb{Z}_{β} assumes infinitely many distinct distances between neighbors.

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Suggestion: Start with the existence of $\lim_{n\to\infty} \frac{b_n}{n}$.

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