Probabilistic aspects of beta-integers viewed as perturbations of integers

Jean-Pierre Gazeau

Laboratoire Astroparticules et Cosmologie
CNRS–Université Paris Diderot Paris 7
gazeau@apc.univ-paris7.fr

A work in progress with
L. Balková (Paris-Prague), and K. Sekimoto (Paris)

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Suppose we are given an infinite strictly increasing sequence of nonnegative real numbers $\{x_n\}_{n\in\mathbb{N}}$ such that $x_0 = 0$, with $x_{n+1} - x_n \geq r > 0$ and $x_{n+1} - x_n \leq R$ for any $n \in \mathbb{N}$ ("Delaunay sequence"). To this sequence of numbers correspond the sequence of "factorials" $x_n! = x_1 x_2 \ldots x_n$ with $x_0! = 1$, the "exponential" $E(t) = \sum_{n=0}^{+\infty} \frac{t^n}{x_n!}$, the sequence of "moment" integrals, $x_n!\mu_n = \int_0^{+\infty} \frac{t^n}{E(t)}\, dt$, and the "renormalized" sequence $\tilde{x}_n := \frac{\mu_n}{\mu_{n-1}} x_n$. In the present study we consider examples of such sequences like non-negative β-integers, with β a Parry number, for which it is proved, analytically or numerically, that $\mu_n \to 1$.

Such sequences of numbers allow to implement[a] a non-commutative reading of the complex plane based on a probabilistic Bayesian content, like that one associated with the standard integers and based on the duality between discrete Poisson distribution and continuous gamma distribution.

[a] *Non-commutative reading of the complex plane through Delaunay sequences* S.T. Ali, L. Balková, E.M.F. Curado, J. P. G, M.A. Rego-Monteiro, Ligia M.C.S. Rodrigues and K. Sekimoto

Delaunay sequences

Consider a strictly increasing sequence of nonnegative real numbers

$$\mathcal{X} = \{x_n\}_{n\in\mathbb{N}}\,, \qquad x_0 = 0\,,$$

with the following two constraints :

(i) $\mathcal{X}$ is *uniformly discrete* on the positive real line $\mathbb{R}^+$: $\exists\, r > 0$ such that $x_{n+1} - x_n \geq r$ for all $n \in N$, which means that there exists a minimal non zero distance between two successive elements of the sequence,

(ii) $\mathcal{X}$ is *relatively dense* on $\mathbb{R}^+$: $\exists\, R > 0$ such that for all $x \in \mathbb{R}^+$ $\exists\, n \in \mathbb{N}$ such that $|x - x_n| < R$, which means that there exists a maximal distance, say L between two successive elements of the sequence.

These conditions imply that $\lim_{n\to\infty} x_n = \infty$. We will denote by $\mathcal{X}$ such a (non-negative) *Delaunay* sequence in $\mathbb{R}^+$.

Comments

- A Delaunay sequence $\mathcal{X}$ should appear as not very different from the set of natural numbers.
- In this regard, sequences like $x_n = n^\alpha$, $\alpha \neq 1$, or $x_n = n \log n$ are not Delaunay
- Familiar deformations of integers, like q-deformations or (p, q)-deformations,

$$[n]_{(p,q)} = \frac{p^{-n} - q^n}{p^{-1} - q}\,, \qquad [n]_q = [n]_{(1,q)}\,,$$

are not Delaunay.

- In this sense, these expressions could be viewed as *singular* deformations of the set $\mathbb{N}$.

Factorials, exponentials, moments

To the sequence $\mathcal{X}$ there correspond

- the sequence of "factorials" $x_n! = x_1 x_2 \ldots x_n$ with $x_0! \stackrel{\text{def}}{=} 1$,
- the "exponential"

$$\mathcal{N}(t) = \sum_{n=0}^{+\infty} \frac{t^n}{x_n!}\,,$$

- and the sequence of "moment" integrals

$$x_n!\,\mu_n = \int_0^{+\infty} \frac{t^n}{\mathcal{N}(t)}\,dt\,,$$

- The appearance of the "corrective" factors μ_n's is needed since there is no reason that the Stieltjes moment problem be solved for a generic pair $(dt/\mathcal{N}(t), x_n!)$ as it would be for $(e^{-t}\,dt, n!)$.

α-Delaunay perturbations of set $\mathbb{N}$

Let $\mathbb{N} \ni n \mapsto \alpha(n)$ be a bounded function with values in the interval $(-1, 1)$ and such that its successive jumps $\alpha(n+1) - \alpha(n)$ have lower bound $r - 1$ with $r \in (0, 1)$. Then the Delaunay sequence

$$x_n = n + \alpha(n)\,, \quad n \in \mathbb{N}\,,$$

is defined as an α-Delaunay perturbation of the natural numbers.
Note that this r is a Delaunay lower bound for the sequence $\{x_n\}_{n \in \mathbb{N}}$

Examples

- For instance, $\alpha(n)$ could be a constant shift, $\alpha(n) = \epsilon$. In order to fulfill the Delaunay condition (i), we should have $-1 < \epsilon < 1$ and $0 < r < 1$.
- It is a particular case of the less trivial example

$$\alpha(n) = \frac{an + b}{cn + d}, \quad ad - bc \neq 0, \quad 0 < r < 1\,.$$

- The perturbation function could also be periodic, like

$$\alpha(n) = \epsilon \sin \omega n\,, \quad -1 < \epsilon < 1\,, \quad 0 < r \leq 1 - 2|\epsilon \sin \omega|\,.$$

- The function $\alpha(n)$ could be a random perturbation.

Beta-integers

- Quite interesting examples are found within the context of numeration systems like sets $\mathbb{Z}_\beta^+$ of non-negative beta-integers b_n, i.e. all these positive real numbers $b_n$which are polynomial in powers of a irrational real number $\beta > 1$ when they are written in "basis" β with the usual greedy algorithm.
- When β is endowed with specific properties (e.g. Pisot algebraic integers), these beta-integers then form a quasiperiodic sequence with a finite number of possible adjacent differences $b_{n+1} - b_n$.
- The simplest example is afforded by the set $\mathbb{Z}_\tau^+$ of non-negative tau-integers, where β is equal to the golden mean $\tau = (1+\sqrt{5})/2$. These tau-integers form a quasiperiodic sequence with two possible adjacent differences $b_{n+1}-b_n = 1$ or $1/\tau$.
- There exists a scaling factor $0 < c_\tau < 1$ and a bounded function $\alpha_\tau(n)$ such that the rescaled sequence of the $x_n \stackrel{\text{def}}{=} b_n/c_\tau = n + \alpha_\tau(n)$ is an α-perturbation of $\mathbb{N}$.

Tau-integers

- Their exact expression is given by

$$\mathbb{Z}_\tau^+ = \left\{ b_n = c_\tau\, n - \frac{1}{\tau^4} + \frac{1}{\tau^2} \left\{ \frac{n+1}{\tau^2} \right\},\ n \in \mathbb{N} \right\}, \quad c_\tau = \frac{1+\tau^2}{\tau^3} \approx 0.8541\,,$$

where $\{x\}$ designates the fractional part of a nonnegative real number x.

- Dividing by c_τ gives the "normalized" Delaunay sequence $x_n = b_n/c_\tau = n + \alpha_\tau(n)$, with

$$\alpha_\tau(n) = \frac{\tau}{1+\tau^2} \left\{ \frac{n+1}{\tau^2} \right\} - \frac{1}{\tau(1+\tau^2)}\,.$$

Due to $\{x\} \in [0,1)$, we observe that the bounds of the fluctuation :

$$-\frac{1}{\tau(1+\tau^2)} \approx -0.1708 < \alpha_\tau(n) < \frac{1}{(1+\tau^2)} \approx 0.2764\,.$$

Beta-integers : main definitions

- A β-*representation* on an alphabet of digits D of a number x of $[0,1]$ is an infinite sequence $(d_j)_{j\geq 1}$ of $D^{\mathbb{N}}$ such that $\sum_{j\geq 1} d_j \beta^{-j} = x$.
- For $x \in [0,1]$ denote $x_1 = \lfloor \beta x \rfloor$ and let $r_1 = \{\beta x\}$. Then iterate for $j \geq 2$, $x_j = \lfloor \beta r_{j-1} \rfloor$ and $r_j = \{\beta r_{j-1}\}$.
- Greedy algorithm : $x = \sum_{j\geq 1} x_j \beta^{-j}$, where the digits x_j are elements of the *canonical* alphabet $A_\beta = \{0, \ldots, \lfloor \beta \rfloor\}$ if $\beta \notin \mathbb{N}$, $A_\beta = \{0, \ldots, \beta - 1\}$ otherwise. The sequence $(x_j)_{j\geq 1}$ of $A_\beta^{\mathbb{N}}$ is the β-*expansion* of x
- The β-expansion obtained by the greedy algorithm is the greatest one in the lexicographic order.
- Let $d_\beta(1) = (t_j)_{j\geq 1}$ be the β-expansion of 1. If $d_\beta(1)$ is finite, $d_\beta(1) = t_1 \cdots t_N$, set $d^*_\beta(1) = (t_1 \cdots t_{N-1}(t_N - 1))^\omega$, otherwise set $d^*_\beta(1) = d_\beta(1)$.
- An infinite word $s = (s_j)_{j\geq 1}$ is the β-expansion of a number x of $[0,1[$ if and only if for every $p \geq 1$, $s_p s_{p+1} \cdots$ is smaller in the lexicographic order than $d^*_\beta(1)$.
- A number β such that $d_\beta(1)$ is eventually periodic is called a *Parry number*.

Results (Balková's talk)

The similarity between sets $\mathbb{N}$ and $\mathbb{Z}_\beta^+ = \{b_n \mid n \in \mathbb{N}\}$ for β being a Parry number is illustrated by the two properties [a] :

- the limit $c_\beta = \lim_{n\to\infty} \frac{b_n}{n}$ exists, i.e. $\mathbb{Z}_\beta^+$ admits *an average structure*
- If $p(x)$ is the Parry polynomial of a simple Parry number β. Then

$$c_\beta := \lim_{n\to\infty} \frac{b_n}{n} = \frac{\beta - 1}{\beta^m - 1} p'(\beta).$$

- If $p(x)$ is the Parry polynomial of the non-simple Parry number β. Then

$$c_\beta := \lim_{n\to\infty} \frac{b_n}{n} = \frac{\beta - 1}{\beta^m(\beta^p - 1)} p'(\beta).$$

- For β being moreover a Pisot-Vijayaraghavan number with mutually distinct roots of its Parry polynomial, it is proven that $(b_n - c_\beta\, n)_{n\in\mathbb{N}}$ is a bounded sequence, i.e. admits an *average structure with bounded modulation*.

[a] L. Balková, JPG, E. Pelantová, Lett. Math. Phys., to appear

Averaged beta-integers

- Suppose now the limit $c_\beta = \lim_{n\to\infty} \dfrac{b_n}{n}$ exists, i.e. $\mathbb{Z}_\beta^+$ admits *an average structure*. Then define the associated Delaunay sequence $\mathcal{X}_\beta$

$$x_n \stackrel{\text{def}}{=} \frac{b_n}{c_\beta}\,.$$

- Further, suppose that β is such that the sequence $\mathcal{X}_\beta$ is a α-Delaunay perturbation of the natural numbers, i.e.,

$$x_n = n + \alpha(n)\,, \quad n \in \mathbb{N}\,,$$

  with $\alpha(n)$ bounded function with values in the interval $(-1, 1)$ and such that its successive jumps $\alpha(n+1) - \alpha(n)$ have lower bound $r - 1$ with $r \in (0, 1)$ (these conditions insure preservation of the order by the one-to-one map $n \mapsto x_n$).

- Question : for which class of numbers β are those conditions fulfilled ?

A Poisson-like discrete distribution issued from $\mathcal{X}_\beta$

- Consider the discrete probability distribution with parameter $t \geq 0$:

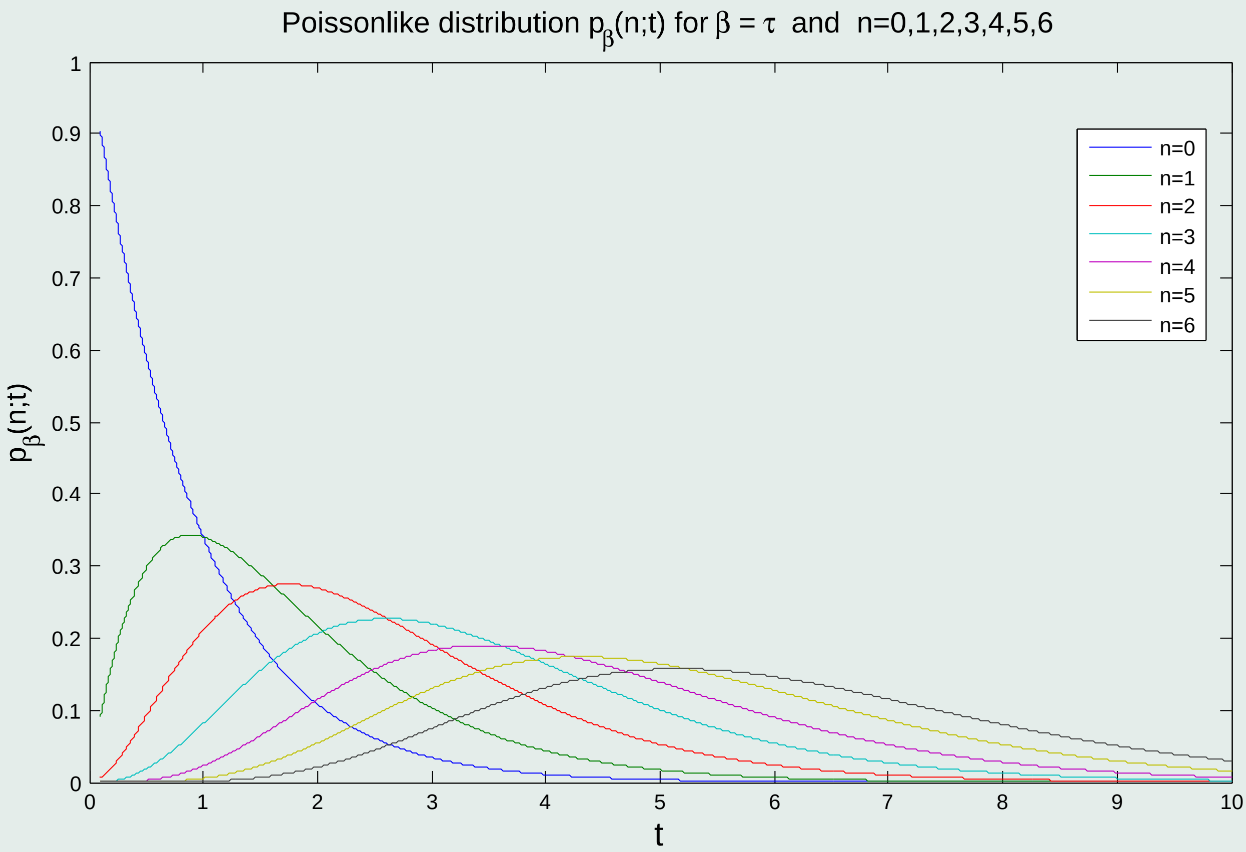
$$n \mapsto p_\beta(n;t) = \frac{1}{\mathcal{N}(t)} \frac{t^n}{x_n!} \,,$$

with $x_n! = x_1 x_2 \cdots x_n$, $x_0! = 1$, and $\mathcal{N}(t) = \sum_{n\geq 0} \frac{t^n}{x_n!}$.

- The average of the random variable $n \mapsto x_n$ is $\langle x_n \rangle = t$.
- Contrariwise to the standard case $\mathcal{X} = \mathbb{N}$, the continuous (gammalike) distribution $t \mapsto \frac{1}{\mathcal{N}(t)} \frac{t^n}{x_n!}$ with parameter n is not a probability distribution with respect to the Lebesgue measure dt :

$$\int_0^{+\infty} \frac{dt}{\mathcal{N}(t)} \frac{t^n}{x_n!} \stackrel{\text{def}}{=} \mu_n \neq 1 \,.$$

Finding the right measure amounts to solve a usually intractable moment problem.



Poissonlike distribution $p_\beta(n;t)$ for $\beta = \tau$ and n=0,1,2,3,4,5,6
$p_\beta(n;t)$
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1
2
3
4
5
6
7
8
9
10
n=0
n=1
n=2
n=3
n=4
n=5
n=6

The moment problem has no solution or has a solution with unsolved measure.

- Suppose that the sequence

$$\mu_n = \int_0^{+\infty} \frac{dt}{\mathcal{N}(t)} \frac{t^n}{x_n!}$$

has a finite limit at $n \to \infty$,

$$\mu_n \to \mu_\infty < \infty \Leftrightarrow \lim_{n\to\infty} \frac{\mu_n}{\mu_{n-1}} = 1\,.$$

- It is then natural to "renormalize" the sequence $\mathcal{X}_\beta = (x_n)$ as follows.

$$\tilde{x}_n \stackrel{\text{def}}{=} \frac{\mu_n}{\mu_{n-1}} x_n\,, \quad n \in \mathbb{N}\,, \quad \Rightarrow \tilde{x}_n! = \mu_n\,(x_n!)\,, \quad n \in \mathbb{N}\,.$$

- Due to $\lim_{n\to\infty} \frac{\mu_n}{\mu_{n-1}} = 1$ this renormalized sequence, say $\widetilde{\mathcal{X}}_\beta$, is closer and closer to the original one, and the strictly increasing order is respected beyond a certain rank n_0.
- Question : for which class of numbers β has the sequence of the μ_n's a finite limit at $n \to \infty$?

Gammalike probability distribution for the renormalized sequence

- Suppose that the sequence $\mu_n = \int_0^{+\infty} \frac{dt}{\mathcal{N}(t)} \frac{t^n}{x_n!}$ has a finite limit μ_∞ at $n \to \infty$. Introduce the renormalized sequence $\tilde{x}_n = \frac{\mu_n}{\mu_{n-1}} x_n$ and the corresponding "exponential"

$$\widetilde{\mathcal{N}}(t) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{t^n}{\tilde{x}_n!} .$$

- Then the map $\mathbb{N} \in n \mapsto \dfrac{t^n}{\widetilde{\mathcal{N}}(t)\tilde{x}_n!}$ is a Poisson-like distribution with average number of occurrences equal to t,
- and the map $0 \leq t \mapsto \dfrac{t^n}{\widetilde{\mathcal{N}}(t)\tilde{x}_n!}$ <u>is</u> a (Gamma-like) probability distribution with $\tilde{x}_{n+1}$ as a shape parameter <u>and</u> with respect to the modified measure

$$dw(t) \stackrel{\text{def}}{=} \frac{\widetilde{\mathcal{N}}(t)}{\mathcal{N}(t)} dt .$$

This last measure should be viewed as a perturbation of the Lebesgue measure.

The existence of μ_∞ for α-Delaunay perturbations of $\mathbb{N}$?

- For an α-Delaunay perturbation $x_n = n + \alpha(n)$ of $\mathbb{N}$ one can write

$$x_n! = n!\,\frac{1}{\xi(n)} \quad \text{with} \quad \frac{1}{\xi(n)} \stackrel{\text{def}}{=} \prod_{k=1}^{n}\left(1 + \frac{\alpha(k)}{k}\right) .$$

- Then the associated "exponential" reads as

$$\mathcal{N}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\xi(n) ,$$

- Its ratio to the ordinary exponential reads as the Poisson average of the random variable $n \mapsto \xi(n)$

$$\mathcal{N}(t)\,e^{-t} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,e^{-t}\xi(n) \stackrel{\text{def}}{=} (\mathrm{E}_P\xi)(t) ,$$

- Thus, the ratio $\mu_n/\xi(n)$ can be rewritten as the gamma average of the random variable $t \mapsto e^t/\mathcal{N}(t)$:

$$\frac{\mu_n}{\xi(n)} = \frac{1}{\xi(n)} \int_0^{+\infty} \frac{dt}{\mathcal{N}(t)}\,\frac{t^n}{x_n!} = \int_0^{\infty} \frac{t^n}{n!}\,e^{-t}\,\frac{e^t}{\mathcal{N}(t)}\,dt \stackrel{\text{def}}{=} \left(\mathrm{E}_G \frac{1}{(\mathrm{E}_P\xi)(t)}\right)(n) .$$

A general result on asymptotic Poisson and Gamma distributions

- Let $\mathbb{N} \ni n \mapsto \xi(n) \in [0,1)$ a discrete function which is extendable to a function $\mathbb{R}^+ \ni x \mapsto [0,1)$ with $\xi(0) = 1$ and $\lim_{x\to\infty} \xi(x) = 0$. Its Poisson mean value with parameter t is defined by

$$(\mathrm{E}_P\xi)(t) := \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{-t}\xi(n)\,,$$

whereas the gamma mean value with parameter n of a random variable Ξ is given by

$$(\mathrm{E}_G\Xi)(n) := \int_0^{\infty} \frac{t^n}{n!} e^{-t}\Xi(t)\,dt\,.$$

- Let us examine the asymptotic behavior of the following combination (μ_n, precisely) of these two averages :

$$\mu_n = \xi(n)\left(\mathrm{E}_G\frac{1}{(\mathrm{E}_P\xi)}\right)(n) = \int_0^{\infty} \frac{\frac{t^n}{n!}\,\xi(n)}{\sum_{m=0}^{\infty}\frac{t^m}{m!}\,\xi(m)}\,dt\,.$$

A sufficient condition for renormalizing a α-Delaunay perturbation of $\mathbb{N}$

Let us define the logarithm of $1/\xi$ by $\xi(x) = e^{-\phi(x)}$, and suppose that the function $\mathbb{R}^+ \ni x \mapsto \phi(x)$ has the following properties.

(i) $\phi(x)$ is twice derivable almost everywhere (a.e.) on $\mathbb{R}^+$,

(ii) $|\phi'(x)| \ll 1$(a.e.) at large x,

(iii) there exists $0 \le \delta < 1$ such that $x|\phi''(x)| \sim \delta$(a.e.) at large x.

Then [a]

$$\mu_n \underset{n\to\infty}{\longrightarrow} 1\,.$$

[a] JPG and K. Sekimoto

Application to α-Delaunay perturbations of $\mathbb{N}$

- Let $x_n = n + \alpha(n)\,,\ n \in \mathbb{N}\,,$ a α-Delaunay perturbation of $\mathbb{N}$ such that the map $\mathbb{N} \ni n \mapsto \alpha(n)$ extends to a function $\mathbb{R}^+ \ni x \mapsto \alpha(x)$ that is a.e. derivable.
- Let us extend the discrete domain of the logarithm $\phi(n) \stackrel{\text{def}}{=} \sum_{k=1}^{n} \ln\left(1 + \frac{\alpha(k)}{k}\right)$ of $1/\xi$ to a continuous one by replacing the sum by the integral

$$\phi(x) \approx \int_1^x \ln\left(1 + \frac{\alpha(y)}{y}\right) dy\,.$$

- Hence,

$$\phi'(x) \approx \ln\left(1 + \frac{\alpha(x)}{x}\right) \underset{\text{at large } x}{\approx} \frac{\alpha(x)}{x}$$

and so $\phi'(x) \to_{x\to\infty} 0$ since $x \mapsto \alpha(x)$ is bounded.
- For the second derivative, we have

$$\phi''(x) \approx \frac{x\alpha'(x) - \alpha(x)}{x^2 + x\alpha(x)} \underset{\text{at large } x}{\approx} \frac{\alpha'(x)}{x}\,.$$

- Suppose that $|\alpha'(x)|$ goes a.e. toward $\delta \in [0,1)$ at large x. Then we have

$$\mu_n = \int_0^\infty \frac{\frac{t^n}{x_n!}}{\sum_{m=0}^\infty \frac{t^m}{x_m!}}\,dt \underset{n\to\infty}{\to} 1 = \mu_\infty\,.$$

An example of such sequences : "Fibonacci chains"

$$\mathbb{R}_*^+ \ni x \mapsto \alpha(x) = \lambda\{\mu x\} + \nu\,,$$

with $|\lambda| < \frac{r-1}{2}$, $r \in (0,1)$, $|\alpha'(x)| = |\lambda\mu| < 1$, and

$$\begin{array}{l} -1 < \nu < 1 - \lambda \ \text{ if } \ \lambda > 0\,, \\ -\lambda - 1 < \nu < 1 \ \text{ if } \ \lambda < 0\,. \end{array}$$

Question (in order to insure the limit $\mu_n \to 1$)

To which class of numbers $\beta > 1$ are the following conditions fulfilled ?

(i) The limit $c_\beta = \lim_{n\to\infty} \dfrac{b_n}{n}$ exists, i.e. $\mathbb{Z}_\beta^+$ admits *an average structure*.

(ii) The associated sequence $\mathcal{X}_\beta$

$$\mathcal{X}_\beta \ni x_n \stackrel{\text{def}}{=} \frac{b_n}{c_\beta}\,.$$

is a α-Delaunay perturbation of the natural numbers, i.e.,

$$x_n = n + \alpha(n)\,, \quad n \in \mathbb{N}\,,$$

with $\alpha(n)$ bounded function with values in the interval $(-1, 1)$ and such that its successive jumps $\alpha(n+1) - \alpha(n)$ have lower bound $r - 1$ with $r \in (0, 1)$ (these conditions insure preservation of the order by the one-to-one map $n \mapsto x_n$).

(iii) The map $\mathbb{N} \ni n \mapsto \alpha(n)$ extends to a function $\mathbb{R}^+ \ni x \mapsto \alpha(x)$ that is a.e. derivable.

(iv) Finally $|\alpha'(x)|$ goes a.e. toward $\delta \in [0, 1)$ at large x.

Complement

Recall : Poisson distribution as a limit of the binomial distribution

It is well known that the Poisson distribution $\mathbb{N} \ni k \mapsto e^{-t}t^k/k!$ with parameter t is the limit at large n of the Bernouilli process of a sequence of n trials with two possible issues, one ("win" say) with probability $p = t/n$, the other one ("loss") with probability $q = 1 - p$. The probability to get k wins after n trials is given by the binomial distribution :

$$p_k^{(n)} = \binom{n}{k} p^k (1-p)^{n-k},$$

and its Poisson limit

$$p_k^{(n)}|_{p=t/n} \underset{n\to\infty}{\longrightarrow} e^{-t}\frac{t^k}{k!}$$

is readily obtained.

A statistical perturbation of the Bernoulli process

- Imagine a sequence of n trials for which the probability to obtain k wins is given by following "perturbation" of the binomial distribution :

$$\mathfrak{p}_k^{(n)} = \frac{x_n!}{x_{n-k}!\, x_k!}\, p^k\, q_{n;k}(p)\,, \qquad \sum_{k=0}^{n} \mathfrak{p}_k^{(n)}(p) = 1\,,$$

where $\{x_n,\, n \in \mathbb{N}\}$ is a Delaunay α-perturbation of natural numbers.

- Here, the sequence of the unknown functions $q_{n;k}(p)$ should be such that, at the limit $n \to \infty$ with $p = t/x_n$, we get our deformation of the Poisson law :

$$\mathfrak{p}_k^{(n)} = \frac{x_n!}{x_{n-k}!\, x_k!} \left(\frac{t}{x_n}\right)^k q_{n;k}\left(\frac{t}{x_n}\right) \underset{n\to\infty}{\to} \frac{1}{\mathcal{N}(t)} \frac{t^k}{x_k!}\,.$$

- In order to have this limit, we first observe with $p = t/x_n$ that for an α-Delaunay perturbation

$$\frac{x_n!}{x_{n-k}!\, x_k!}\, p^k\, q_{n;k}(p) = \frac{x_n}{x_n} \frac{x_{n-1}}{x_n} \cdots \frac{x_{n-k+1}}{x_n} \frac{t^k}{x_k!}\, q_{n;k}\left(\frac{t}{x_n}\right) \underset{n\to\infty}{\to} \frac{t^k}{x_k!} \lim_{n\to\infty} q_{n;k}\left(\frac{t}{x_n}\right)\,,$$

- So we should have

$$\lim_{n\to\infty} q_{n;k}\left(\frac{t}{x_n}\right) = \frac{1}{\mathcal{N}(t)}\,.$$