Dynamical aspects of β-expansions with deleted digits

Charlene Kalle,
joint work with Karma Dajani

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Introduction

Let $\beta > 1$ and $A = \{a_0, \ldots, a_m\}$ a set of real numbers. Expansions of the form

$$x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i},$$

with $b_i \in A$ for all $i \geq 1$, are called *β-expansions with deleted digits.*

This gives numbers in the interval $\left[\frac{a_0}{\beta - 1}, \frac{a_m}{\beta - 1}\right]$.

Allowable digit sets

If, for a given $\beta > 1$, a set of real numbers $A = \{a_0, \dots, a_m\}$ satisfies

(i) $a_0 < \dots < a_m$,

(ii) $\max_{1 \le j \le m}(a_j - a_{j-1}) \le \frac{a_m - a_0}{\beta - 1}$,

it is called an *allowable digit set*. Then

- every $x \in \left[\frac{a_0}{\beta - 1}, \frac{a_m}{\beta - 1}\right]$ has a β-expansion with deleted digits. (Pedicini, 2005)
- the minimal amount of digits in A is $\lceil \beta \rceil$.

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The classical β-expansion

Take $\beta > 1$ and $A = \{0, 1, \ldots, \lfloor \beta \rfloor\}$. This gives the classical β-expansions.

- We can use the greedy β-transformation to generate such expansions by iteration.
- The greedy β-transformation has an invariant measure that is equivalent to the Lebesgue measure on the unit interval $[0, 1)$. (Rényi, 1957)

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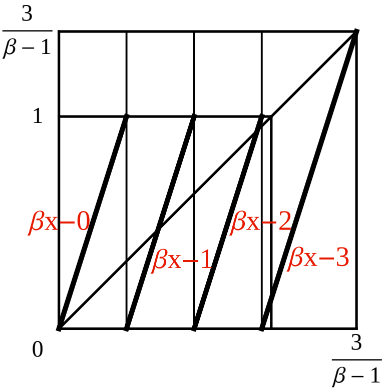
The classical greedy β-transformation

The classical greedy β-transformation is given by

$$Tx = \begin{cases} \beta x - i, & \text{if } x \in \left[\frac{i}{\beta}, \frac{i+1}{\beta}\right),\ i \in \{0, \dots, \lfloor\beta\rfloor - 1\}, \\ \beta x - \lfloor\beta\rfloor, & \text{if } x \in \left[\frac{\lfloor\beta\rfloor}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right]. \end{cases}$$



The classical greedy β-transformation

The transformation T has the following properties.

- The support of the invariant measure, absolutely continuous wrt the Lebesgue measure is the interval $[0, 1)$.
- The density function is given by

$$h_c : [0,1) \to [0,1) : x \mapsto \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{1}{\beta^n} 1_{[0,T^n 1)}(x),$$

where $F(\beta) = \int_0^1 \sum_{x<T^n 1} \frac{1}{\beta^n} d\lambda$ is a normalizing constant and λ is the Lebesgue measure. (Gel'fond, 1959, and Parry, 1960)

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Greedy expansions

There is a recursive algorithm that generates so called *greedy β-expansions with deleted digits*. (Pedicini, 2005)

If the first $n-1$ digits, $b_1, \ldots, b_{n-1}$, of the expansion of x are already known, then the n-th digit is the largest element of A, such that

$$\sum_{i=1}^{n-1} \frac{b_i}{\beta^i} + \frac{b_n}{\beta^n} + \sum_{i=n+1}^{\infty} \frac{a_0}{\beta^i} \leq x.$$

Preserving the Lexicographical Ordering

Theorem (Pedicini, 2005)

Let $<_{lex}$ denote the lexicographical ordering on the set of sequences. If $x = \sum_{i=1}^{\infty} \frac{b_i}{\beta^i}$ and $y = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}$ are the greedy expansions of x and y in base β and digits in an allowable digit set A, then

$$x < y \Leftrightarrow (b_1, b_2, \ldots) <_{lex} (d_1, d_2, \ldots).$$

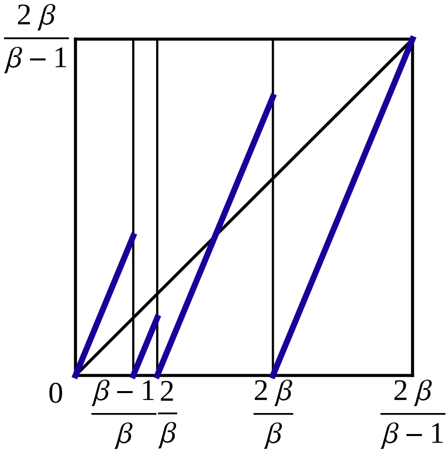
Definition of the greedy β-transformation with deleted digits

Suppose $\beta > 1$ and $A = \{a_0, a_1, \ldots, a_m\}$ is an allowable digit set with $a_0 = 0$. The following transformation generates greedy β-expansions with digits in the set A.

$$Tx = \begin{cases} \beta x - a_j, & \text{if } x \in \left[\dfrac{a_j}{\beta}, \dfrac{a_{j+1}}{\beta}\right), \\ & \text{for } j = 0, \ldots, m-1, \\ \beta x - a_m, & \text{if } x \in \left[\dfrac{a_m}{\beta}, \dfrac{a_m}{\beta - 1}\right]. \end{cases}$$

We can define a similar transformation for digit sets not starting with 0, but that transformation is isomorphic to T.

$\beta = 1 + \sqrt{2}$ and $A = \{0, \beta - 1, 2, 2\beta\}$



Defining the digit sequence

The digit sequence $\{b_n\}_{n=1}^{\infty}$ is given by

$$b_1 = b_1(x) = \begin{cases} a_j, & \text{if } x \in \left[\frac{a_j}{\beta}, \frac{a_{j+1}}{\beta}\right), \\ & \text{for } j = 0, \ldots, m-1, \\ \\ a_m, & \text{if } x \in \left[\frac{a_m}{\beta}, \frac{a_m}{\beta - 1}\right], \end{cases}$$

and $b_n = b_n(x) = b_1(T^{n-1}x)$. Then

$$x = \sum_{n=1}^{\infty} \frac{b_n}{\beta^n}$$

is the greedy β-expansion with deleted digits of x.

The Invariant Measure

By results from Li and Yorke (1978),

- There exists a unique invariant measure for T that is absolutely continuous with respect to Lebesgue.
- This measure is ergodic.
- The support of this measure is an interval of the form $[0, s)$ for some $s \leq \frac{a_m}{\beta-1}$.

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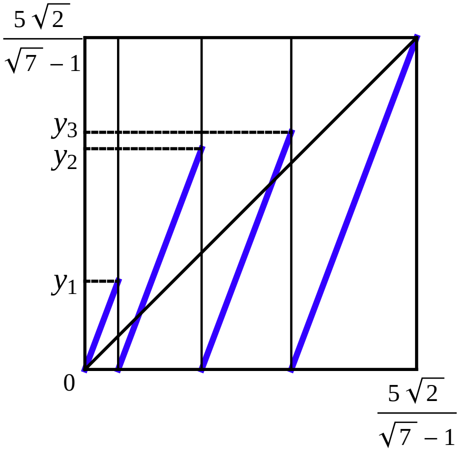
The support of the invariant measure

Consider the points of discontinuity of T, $\frac{a_i}{\beta}$ for $1 \leq i \leq m$. For each i, let y_i denote the limit from the left to the i-th point, i.e.

$$y_i = \lim_{x \uparrow \frac{a_i}{\beta}} Tx.$$

Then the support is the interval $[0, y_j)$, where j is the smallest index such that $T[0, y_j) \subseteq [0, y_j)$.

$\beta = \sqrt{7}$ and $A = \{0, \pi - 1, 4, 5\sqrt{2}\}$



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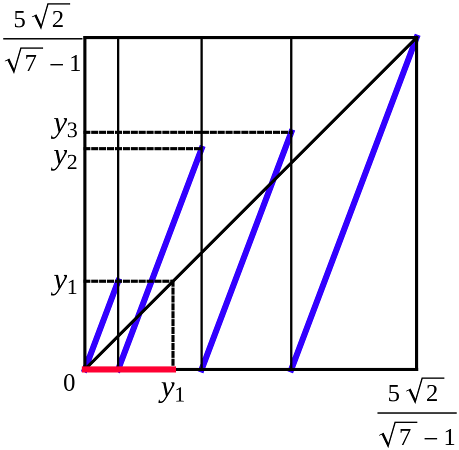

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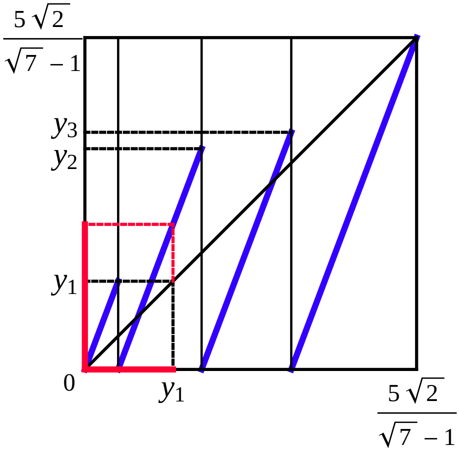

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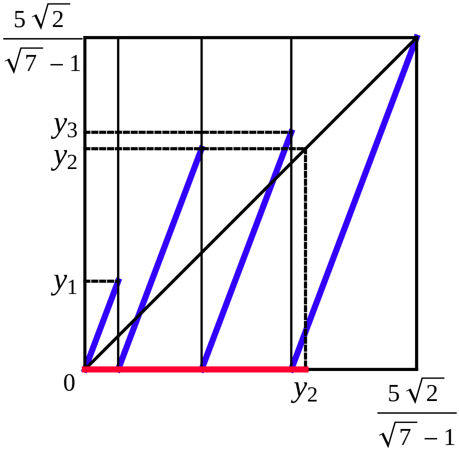

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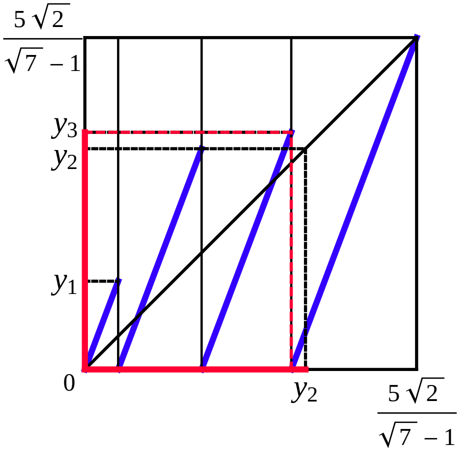

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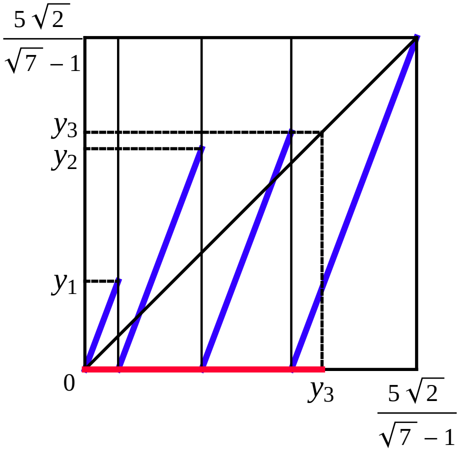

A formula for the density

In some cases we have an explicit expression for the density function.

- If the amount of digits in the digit set is minimal, i.e. if $m < \beta \leq m+1$ and $A = \{0, a_1, \ldots, a_m\}$.
- If the digit set contains 3 digits.
- If the endpoints y_j have ultimately periodic orbits, then the system can be described by a Markov chain.

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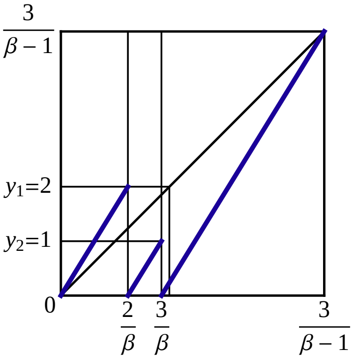
Example of a density function

Let $\beta = \dfrac{1+\sqrt{5}}{2}$ and $A = \{0, 2, 3\}$. The greedy β-transformation with deleted digits is:



The support is $[0, 2)$.

Example of a density function

The density function contains the orbits of y_1 and y_2, which are periodic:

$$\begin{array}{ccccccc} 1 & \rightarrow & \beta & \rightarrow & \frac{1}{\beta} & \rightarrow & 1 \\ 2 & \rightarrow & \frac{1}{\beta^3} & \rightarrow & \frac{1}{\beta^2} & \rightarrow & \frac{1}{\beta} \end{array}$$

The density function is given by

$$\begin{aligned} h(x) &= \frac{1}{16-7\beta}\Big[\, 1_{[0,2)}(x) + 1_{[0,1)}(x) + \frac{1}{\beta}1_{[0,T2)}(x) + \frac{1}{\beta}1_{[0,T1)}(x) \\ &\quad + \frac{1}{\beta^2}1_{[0,T^2 2)}(x) + \frac{1}{\beta}1_{[0,T^2 1)}(x)\,\Big]. \end{aligned}$$

Choosing the smallest digit possible

We can define another algorithm, the *lazy algorithm*, recursively as follows: If $c_1, \ldots, c_{n-1}$ are already given, then c_n is the smallest element of A, such that

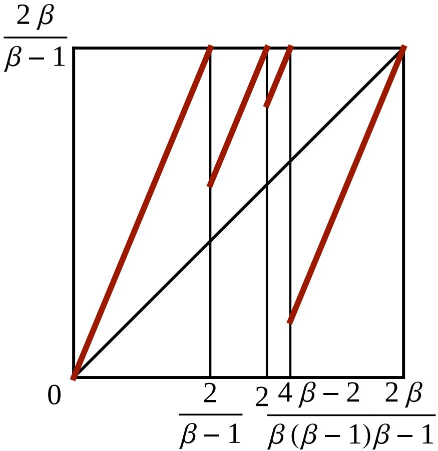
$$x \leq \sum_{i=1}^{n-1} \frac{c_i}{\beta^i} + \frac{c_n}{\beta^n} + \sum_{i=n+1}^{\infty} \frac{a_m}{\beta^i}.$$

Dynamical definition of the lazy transformation

This recursion leads to the following definition:

$$Lx = \begin{cases} \beta x - a_0, & \text{if } x \in \left[\dfrac{a_0}{\beta-1}, \dfrac{a_m}{\beta-1} - \dfrac{a_m - a_0}{\beta}\right], \\ \beta x - a_j, & \text{if } x \in \left(\dfrac{a_m}{\beta-1} - \dfrac{a_m - a_{j-1}}{\beta}, \dfrac{a_m}{\beta-1} - \dfrac{a_m - a_j}{\beta}\right], \\ & \text{for } j = 1, \dots, m. \end{cases}$$

$\beta = 1 + \sqrt{2}$ and $A = \{0, \beta - 1, 2, 2\beta\}$

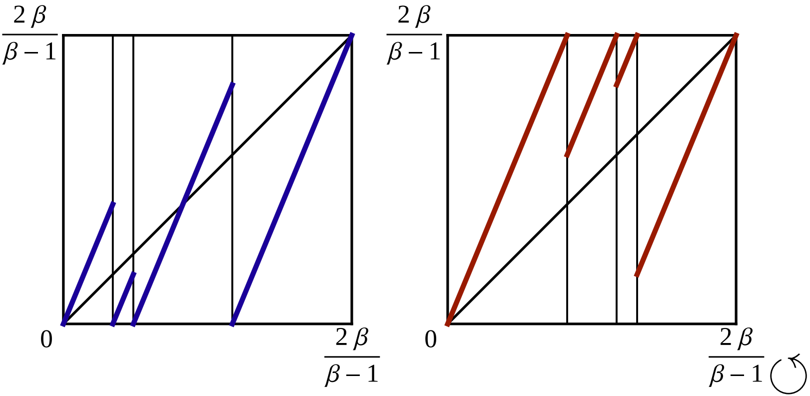


Isomorphic transformations.

There exists an isomorphism between the greedy transformation for some β and $A = \{0, a_1, \ldots, a_m\}$ and the lazy transformation for the same β, but with digit set $\overline{A} = \{\overline{a}_m, \ldots, \overline{a}_0\}$, where $\overline{a}_i = a_m - a_i$. The isomorphism is given by

$$\phi(x) = \frac{a_m}{\beta - 1} - x.$$

The isomorphism between the two transformations



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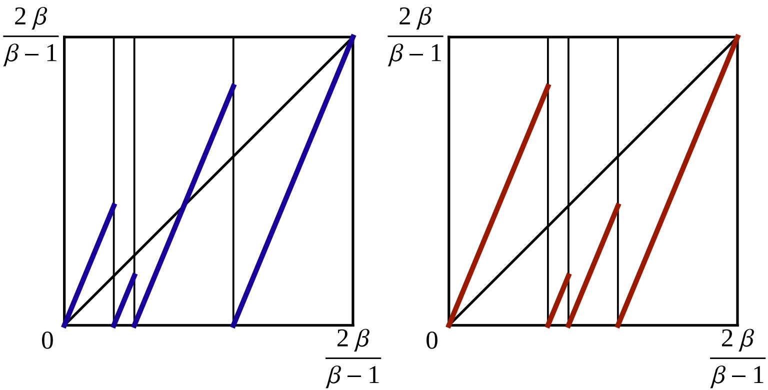
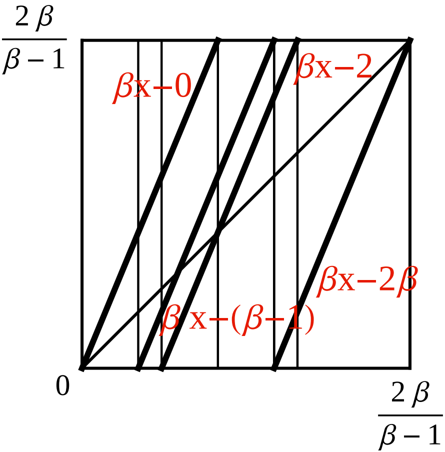


Invariant measure for the lazy transformation

Due to the isomorphism, there also exists a unique invariant measure for L that is absolutely continuous with respect to the Lebesgue measure and is ergodic. (By the same results from Li and Yorke, 1978)

Two extreme cases: $\beta = 1 + \sqrt{2}$ and $A = \{0, \beta - 1, 2, 2\beta\}$

The greedy and lazy transformation are not the only transformations that generate β-expansions with deleted digits, but they are the two extreme cases.



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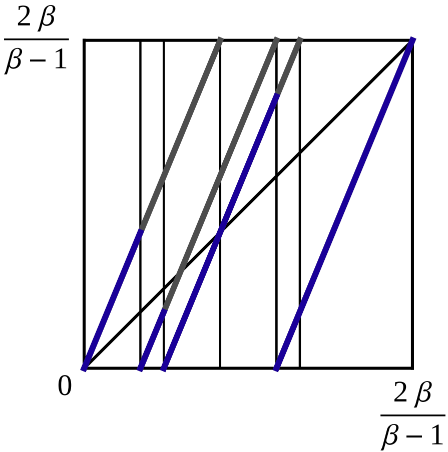

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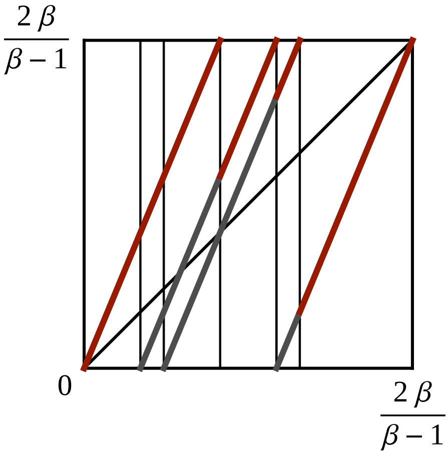

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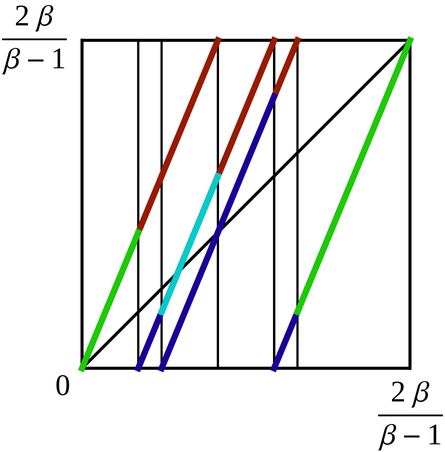


A (β, α)-transformation

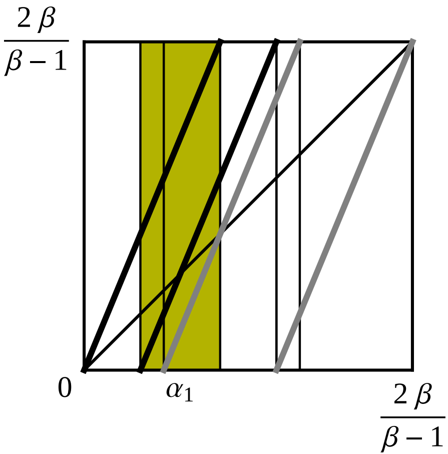
One way of defining another transformation that generates β-expansions with deleted digits is by choosing for each $1 \leq i \leq m$ a value

$$\alpha_i \in \left[\frac{a_i}{\beta}, \frac{a_m}{\beta(\beta-1)} + \frac{a_{i-1}}{\beta}\right]$$

as a point of discontinuity in such a way that

$$\alpha_1 < \alpha_2 < \ldots < \alpha_m.$$

Choosing α's



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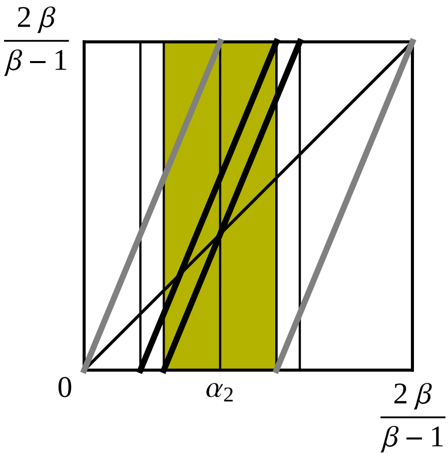

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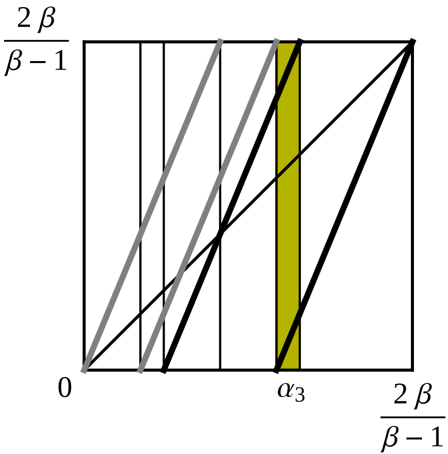
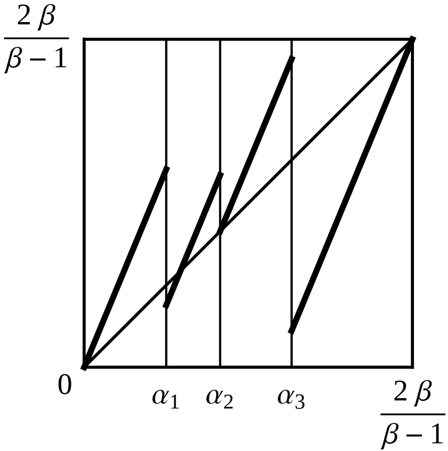


A (β, α)-transformation



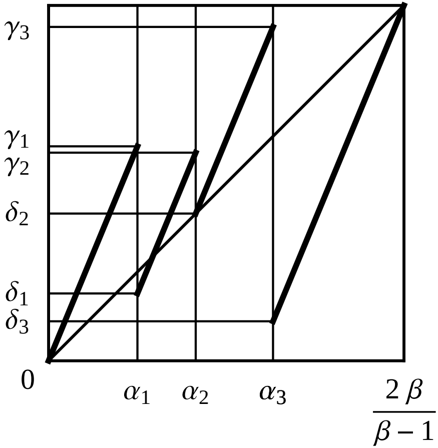
A (β, α)-transformation

Set $\alpha_0 = 0$ and $\alpha_{m+1} = \dfrac{a_m}{\beta - 1}$. The (β, α)-transformation, $T_{(\beta,\alpha)}$, is defined from the interval $\left[0, \dfrac{a_m}{\beta - 1}\right]$ to itself by

$$T_{(\beta,\alpha)}x = \begin{cases} \beta x - a_j, & \text{if } x \in [\alpha_j, \alpha_{j+1}),\ j \in \{0, \ldots, m\}, \\ \dfrac{a_m}{\beta - 1}, & \text{if } x = \dfrac{a_m}{\beta - 1}. \end{cases}$$

A (β, α)-transformation

For each $1 \leq j \leq m$, let δ_j be the limit from the right and γ_j be the limits from the left to α_j. Put $\delta_0 = 0$ and $\gamma_{m+1} = \dfrac{a_m}{\beta - 1}$.



A lexicographical characterization

Let $x = \sum_{n=1}^{\infty} \frac{x_n}{\beta^n}$ be a β-expansion with deleted digits for x and let $\delta_j = \sum_{n=1}^{\infty} \frac{b_n(\delta_j)}{\beta^n}$ and $\gamma_j = \sum_{n=1}^{\infty} \frac{b_n(\gamma_j)}{\beta^n}$ be the expansions generated by the (β, α)-transformation.

Theorem

The expansion of x is the expansion generated by the (β, α)-transformation iff the following condition holds: For all $n \geq 1$, $x_n = a_j$ implies

$$b_1(\delta_j)b_2(\delta_j)\ldots \leq_{\text{lex}} x_{n+1}x_{n+2}\ldots <_{\text{lex}} b_1(\gamma_j)b_2(\gamma_j)\ldots$$