Some dynamical properties of generalized radix sytems

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Notations I.

- Let Λ be a lattice in $\mathbb{R}^n$
- $M : \Lambda \to \Lambda$ such that $\det(M) \neq 0$ (*base*)
- $0 \in D \subseteq \Lambda$ a finite subset (*digit set*)

Definition The triple (Λ, M, D) is called a *number system* (GNS) if every element $x \in \Lambda$ has a unique, finite representation of the form

$$x = \sum_{i=0}^{l} M^i d_i, \quad (d_i \in D,\ l \in \mathbb{N}).$$

Notations II.

Theorem. If (Λ, M, D) is a number system then

1. D must be a full residue system modulo M
2. M must be expansive
3. $\det(I - M) \neq \pm 1$

If a system fulfills conditions $1.$ and $2.$ then it is called a *radix system*.

Notations III.

- Since a basis transformation does not change the number system property therefore it may assume with no loss of generality that M is integral acting on the lattice $\mathbb{Z}^n$.
- Let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ for which $M\varphi(x) - x \in D$ holds. φ is well-defined since D is a full residue system.
- A point $p \in \mathbb{Z}^n$ is called *periodic* if $\phi^k(p) = p$ for some $k > 0$
- $\mathcal{P}$ denotes the set of all periodic elements

Standard expansion

Let $z \in \mathbb{Z}^n$. If $z_0 := z \notin \mathcal{P}$ then there is a unique $l \in \mathbb{N}$ (length of expansion) and $d_0, d_1, \dots, d_{l-1} \in D$ such that

$$z_j = d_j + M z_{j+1} \; (j = 0, \dots, l-1), \;\; z_l = p \in \mathcal{P}$$

and none of $z_0, z_1, \dots, z_{l-1}$ do belong to $\mathcal{P}$. Then the *standard expansion* of z is

$$z = (d_0, d_1, \dots, d_{l-1} \mid p) \,.$$

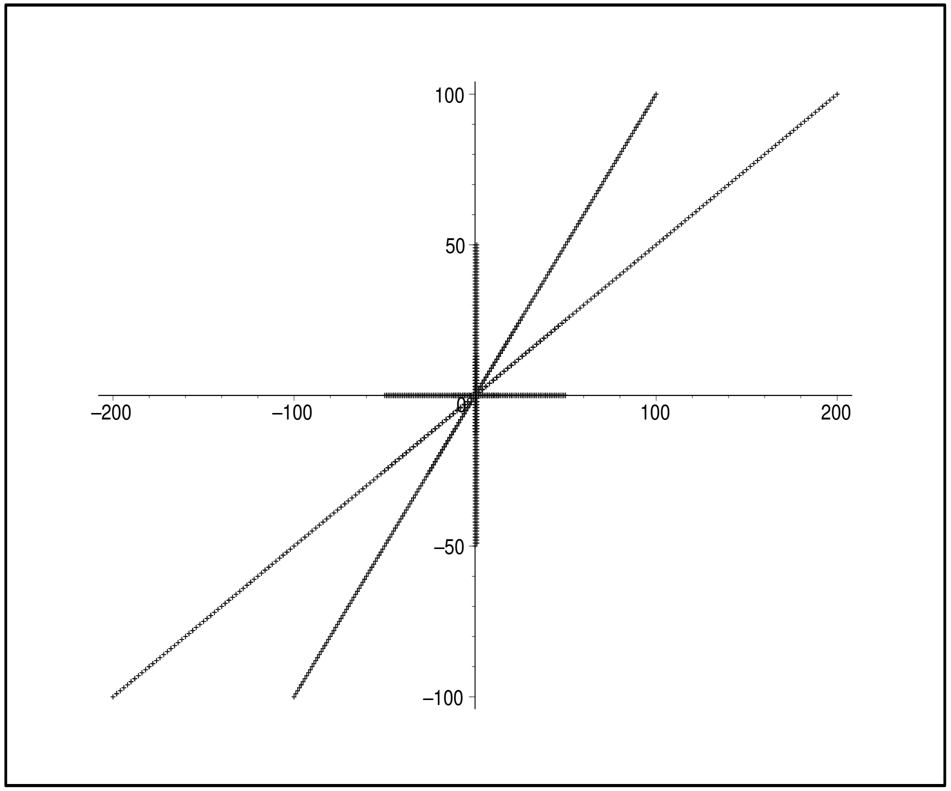
If $p \in \mathcal{P}$ then its standard expansion is $p = (* \mid p)$.

Notations IV.

- Let $S, T \subseteq \mathbb{Z}^n$
- Let $\varphi^+(S) = \bigcup_{k=0}^{\infty} \varphi^k(S) \subseteq \mathbb{Z}^n$ (the orbits of S)
- If $\forall x \in T \;\; \exists k \in \mathbb{N}, s \in S : \varphi^k(s) = x$ then we say that the orbits of S cover T. Notation: $\mathrm{cover}_\varphi(S, T)$
- Let $M^-(S) = \bigcup_{k=0}^{\infty} M^{-k}(S) \subseteq \mathbb{R}^n$ (the right shifts of S)

Example

$M = companion(x^2 + 2x + 2), D = \{0, 1\}$,
$S = \{(i, 0), i = -200..200\}$ The set $\varphi^+(S)$:



Notations V.

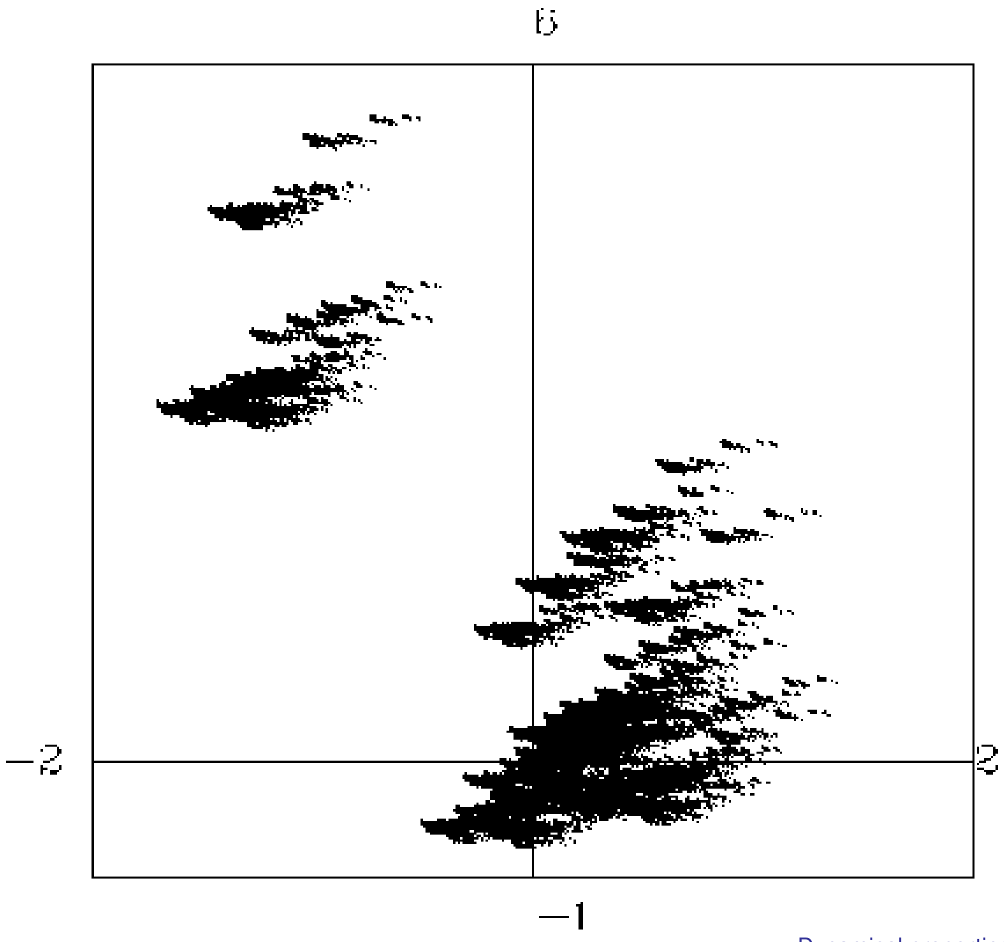
The set of “fractions” in $(\mathbb{Z}^n, M, D)$ is

$$H = \left\{ \sum_{n=1}^{\infty} M^{-n} d_n : d_n \in D \right\} \subseteq \mathbb{R}^k .$$

- Let $\Gamma_l = d_0 + M d_1 + \cdots + M^l d_l,\ (d_i \in D)$.
- Let $\Gamma = \bigcup \Gamma_l$ (all lattice points with finite expansions)

Example

$$M = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, D = \{(0,0), (1,0), (0,1), (0,-1), (-6,-5)\}.$$



Known results

- H is compact.
- H has interior points. More specifically $\bigcup_{p\in\mathcal{P}}(p+H)$ contains a neighborhood of the origin.
- $\mathbb{R}^n = \bigcup(H+\mathbb{Z}^n)$
- For every $x \in \mathbb{R}^n$ there is a $z \in \mathbb{Z}^n$ and $h \in H$ such that $x = z + h$.
- $\lambda(H) > 0$ (λ is the Lebesgue measure on $\mathbb{R}^n$).
- $\lambda(H + z_1 \cap H + z_2) = 0$ for all $z_1 \neq z_2 \in \Gamma$.

Sets with covering orbits

Problem 1.

Find a “small” set S for which

$$\mathrm{cover}_{\varphi}(S, \mathbb{Z}^n)$$

holds.

Motivation

- Computer experiments show that the classification often can be made only by checking the orbits of points of a single axis.
- In these experiments the role of S was played by an axis, a 1-dimensional subspace.
- In dimension 2 several pairs (M, D) was examined, and $\mathrm{cover}_{\varphi}(\mathbb{Z}, \mathbb{Z}^2)$ holds.

Formalization I.

- Every $x \in \mathbb{Z}^n$ can be written in the standard form
$$x = (d_0, d_1, \ldots, d_{l-1} \mid p) \,.$$
where p is a periodic element.
- If $\varphi^k(s) = x$ for some $s \in \mathbb{Z}^n, k \in \mathbb{N}$ then
$$s = (c_0, c_1, \ldots, c_{k-1}, d_0, d_1, \ldots, d_{l-1} \mid p) \,.$$
- Thus, the expansion of s begins with the periodic element and the l digits of the expansion of x.

Formalization II.

- $\mathrm{cover}_\varphi(S, \mathbb{Z}^n) \iff$ specifying a periodic element and a finite number of digits, one can always find $s \in S$ whose expansion begins with the specified period and digits.
- An analogous motivating example for the usual decimal system with $S = \{2^i \mid i \in \mathbb{N}\}$ is the following. Specify an l-digit positive number x. It is easy to show that there is always a power of 2 whose l-digit prefix is x.

Formalization III.

- Reformulation can be given using fractions: $\mathrm{cover}_\varphi(S, \mathbb{Z}^n)$ holds iff $\forall x \in \mathbb{Z}^n$, $\exists s \in S, k \in \mathbb{N}$ with

$$M^{-k}s = x + \sum_{i=0}^{k-1} M^{i-k} c_i \,.$$

- This equation means that $M^{-k}(s) - x$ is a k-digit fraction in H.

Necessary conditions I.

- A necessary condition for $\mathrm{cover}_\varphi(S, \mathbb{Z}^n)$: $\forall x \in \mathbb{Z}^n$, $\exists s \in S, k \in \mathbb{N}$ with

$$M^{-k}(s) \in x + H\,,$$

that is for all x, $M^{-}(S) \cap (x + H) \neq \varnothing$,

- This is not sufficient. In general, $M^{-k}(s) \in x + H$ only implies $\varphi^k(s') = x$ for some $s' \in s + H$ and s' can be outside S. Consider the 1-dimensional example $M = (-7)$, $D = \{0, 2, 4, 6, 8, 10, 12\}$, $S = \Gamma = 2\mathbb{Z}$, $x = 1$.

Necessary conditions II.

- $M^{-}(S)$ has to intersect every translate of H (of the form $x+H$). Can we say more?
- Let $h=\sum_{i=1}^{l} M^{-i}d_i$ be an l-digit fraction, and $x\in\mathbb{Z}^n$. Then $y=M^l(x+h)\in\mathbb{Z}^n$. It is easily seen that $M^{-}(S)\cap(y+H)\neq\varnothing \iff M^{-}(S)\cap\left((x+h)+M^{-l}H\right)\neq\varnothing$. Since M^{-1} is contractive and H is compact, it follows that $M^{-}(S)\cap(x+H)$ is dense in $x+H$.

Necessary conditions III.

We have that

- If $\mathrm{cover}_\varphi(S, \mathbb{Z}^n)$ holds then $M^-(S) \cap \mathbb{Z}^n$ is dense in $\mathbb{Z}^n + H$.
- The complete residue system property of D implies $\mathbb{Z}^n + H = \mathbb{R}^n$.
- This is equivalent to say that $M^-(S) \cap H$ is dense in H.
- The condition is not sufficient, see example above.

The case of number systems I.

- If $(\mathbb{Z}^n, M, D)$ is a number system then it turns out that the denseness of $M^{-}(S)$ is sufficient.

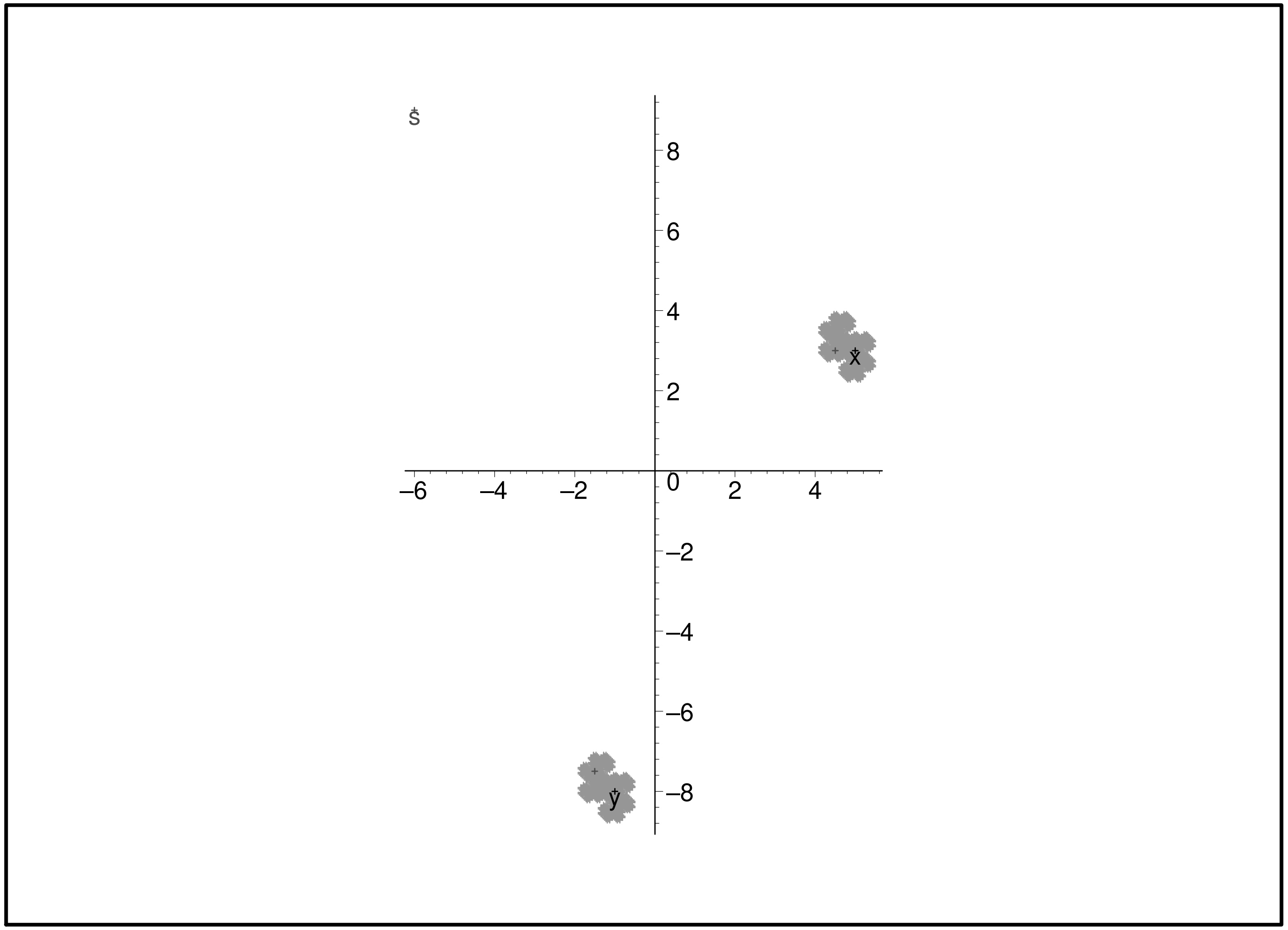
Claim: Let $(\mathbb{Z}^n, M, D)$ be a GNS, $S \subseteq \mathbb{Z}^n$. Then $\mathrm{cover}_{\varphi}(S, \mathbb{Z}^n)$ holds iff $M^{-}(S)$ is dense in $\mathbb{R}^n$.

The case of number systems II.

Proof:

- We will need the following known result: $(\mathbb{Z}^n, M, D)$ is a number system iff 0 is the only lattice point of H.
- By denseness and the nonempty interior of H, we have nonempty intersection of $M^-(S)$ and $x + H$, which yields an $s \in S$ and $s' \in (s + H) \cap \mathbb{Z}^n, k \in \mathbb{N}$ with $\varphi^k(s') = x$. By the number system property, $s' = s \in S$, and we are ready.

The case of number systems III.



Sufficient condition

The sufficient condition can be weakened.

- There is an open ball in H that is disjoint from $H + x$ for $x \in \mathbb{Z}^n \setminus \{0\}$.
- S is dense in $\mathbb{R}^n$.

Note: The first condition has a vast literature, we now briefly discuss denseness.

Dimension 1

In 1 dimension, a few possible S that has dense orbits:

- Even numbers
- Any syndetic sets (sets with bounded gaps)
- Values of a fixed polynomial
- Powers of q, where q is not a rational power of the base.

Dimension 2

In dimension 2, there are two cases to consider:

- Complex case: the eigenvalues of M form a conjugate pair.
- Real case: M has two real eigenvalues (forming further 3 subcases $|\lambda_1| \neq |\lambda_2|$, $|\lambda_1| = |\lambda_2|$ with geometric multiplicity 1 or 2).

Dim 2, complex case

Possible sets S:

- Syndetic sets.
- If the angle of the eigenvalues is not a rational multiple of π then an axis (or even a half axis) covers everything. More generally, lattice points in an arbitrary direction (say, the 1-neighbourhood of a line).
- Lattice points with norm close to some power of q (say, the c-neighbourhoods of circles with radii q^k), where q is not a rational power of the modulus of the dominant eigenvalue.

Dim 2, real case I.

Possible sets S if $|\lambda_1| \neq |\lambda_2|$:

- Syndetic sets.
- Any double cone centered around the dominant eigenvector of M.
- Lattice points with norm close to some power of q, not a rational power of the dominant eigenvector.

Note: No axis is appropriate in this case.

Dim 2, real case II.

Possible sets S if $|\lambda_1| = |\lambda_2|$:

- Lattice points with norm close to some power of q, not a rational power of the eigenvectors.
- A cone orthogonal (in the Jordan base) to the eigenvector if the geometric multiplicity is 1.
- The direct product of 1-dimensional examples if the geometric multiplicity is 2.

Higher Dimensions

- The situation gets more complicated, but syndetic sets and points with norms close to the powers of a well-chosen q will work.
- If there is a single dominant eigenvalue, then a cone around the corresponding eigenvector works.
- No axis suffices for $n \geq 3$.

Example: twin dragon

The non-rationality conditions above are necessary. Consider the base $-1+i$ with canonical digit set in the Gaussian integers. The following sets do not have covering orbits.

- S is an axis (or the union of finitely many directions).
- S is a union of fixed-width annuli around circles of radii 2^k.

Work in progress...

- Extend to other numeration systems: Beta-expansions, SRS, etc.
- Application in coding?
- Closest point on the axis which runs into x
- length of this run.

Length of expansions

The length of the standard expansion of $z \in \mathbb{Z}^n$ can be estimated.
For a given $(\mathbb{Z}^n, M, D)$ and $z \in \mathbb{Z}^n$ there is a constant $c \in \mathbb{N}$ for which

$$l(z) \leq \frac{\log \|z\|}{\log \left(1/\|M^{-1}\|\right)} + c.$$

(The norm means the vector norm and the associated matrix norm for which $\|M^{-1}\| < 1$.)

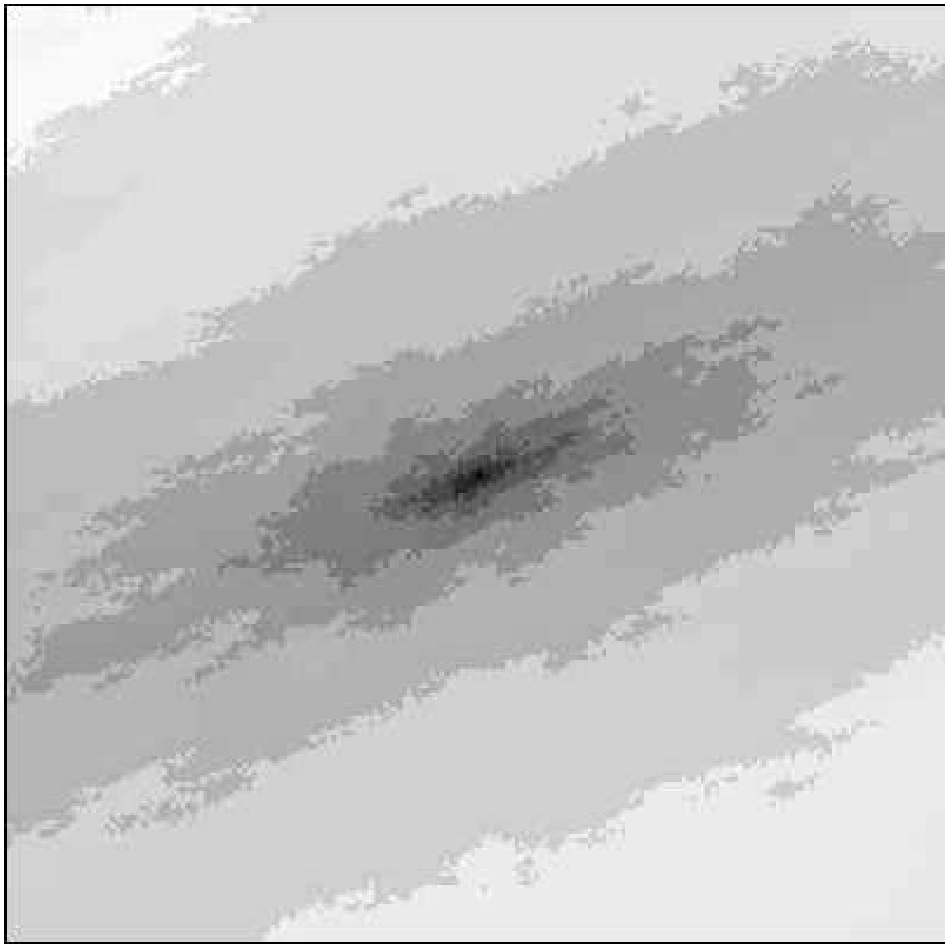
Problem Nr. 2

Problem 2.

What can be stated about the length of expansions for lattice points "close" together?

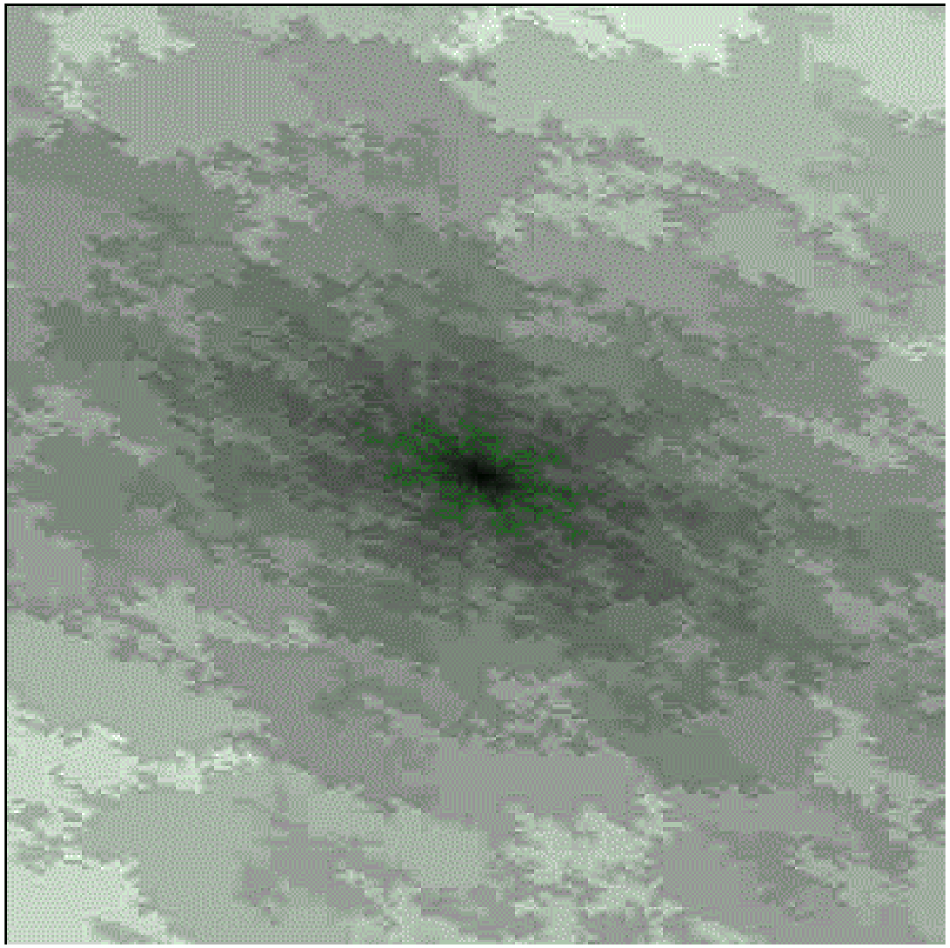
Example I.

$M = companion(x^6 + x^5 - x^3 - x^2 + x + 2),\ D = \{0, 1\}$.



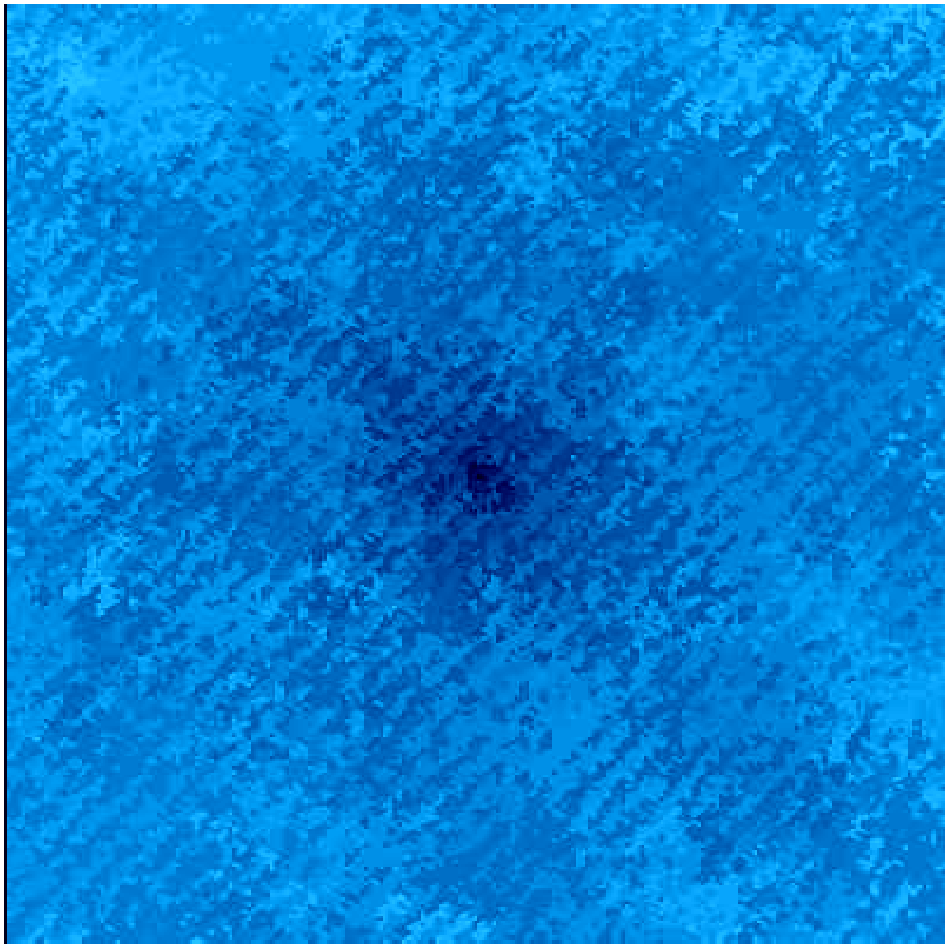
Example II.

$$f = x^6 + 2x^5 + 3x^4 + 3x^3 + 3x^2 + 3x + 2, D = \{0, 1\}.$$



Example III.

$M = companion(x^4+2x^3+2x^2+2x+2),\ D = \{0, 1\}$.



Length of expansions II.

- Application in data compression?

Thank you.