Möbius number systems

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Fractal attractor

contraction: $d(f_a(x), f_a(y)) \leq q \cdot d(x, y),\ q < 1$

Iterative systems

$(F_a : X \to X)_{a \in A}$, X compact, F_a continuous.

$$F_u(x) = F_{u_0} \circ F_{u_1} \circ \cdots \circ F_{u_{n-1}}(x), \; u \in A^n \subset A^+$$

Theorem[Barnsley] If $(F_a : X \to X)_{a \in A}$ are contractions on compact metric X, then there exists a unique attractor $Y = \bigcup_{a \in A} F_a(Y) \subseteq X$.

There exists continuous surjective $\Phi : A^{\mathbb{N}} \to Y$

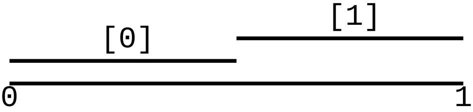
$$\bigcap_{n=0}^{\infty} F_{u_0} \ldots F_{u_{n-1}}(X) = \{\Phi(u)\}, \; u \in A^{\mathbb{N}}$$

Binary system $A = \{0, 1\}$

$$F_0(x) = \frac{x}{2}, \ F_1(x) = \frac{x+1}{2}$$

$$Y = [0,1] = [0, \tfrac{1}{2}] \cup [\tfrac{1}{2}, 1] = F_0(Y) \cup F_1(Y) = [0] \cup [1]$$

$$\Phi_2(u) = \sum_{i \geq 0} u_i \cdot 2^{-i-1}, \ u \in A^{\mathbb{N}}$$

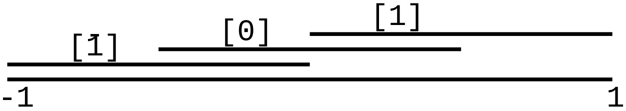


Redundant binary system $A = \{\bar{1}, 0, 1\}$

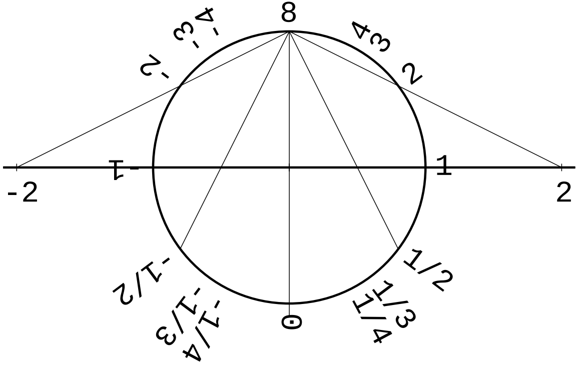
$$F_{\bar{1}}(x) = \frac{x-1}{2},\ F_0(x) = \frac{x}{2},\ F_1(x) = \frac{x+1}{2}$$

$$Y = [-1,1] = [-1,0] \cup [-\tfrac{1}{2}, \tfrac{1}{2}] \cup [0,1] = \bigcup_a F_a(Y)$$

$$\Phi_3(u) = \sum_{i \geq 0} u_i \cdot 2^{-i-1},\ u \in A^{\mathbb{N}}$$



Extended real line: stereographic projection

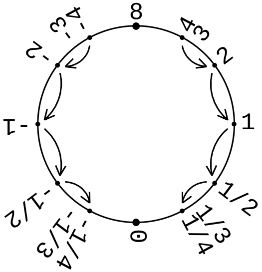


$$\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \;\approx\; \partial\mathbb{D} = \{x \in \mathbb{C} : \; |x| = 1\}$$

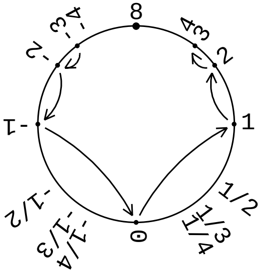
$$\approx \; \mathbb{T} = [-\pi, \pi)$$

Orientation preserving Möbius transformations

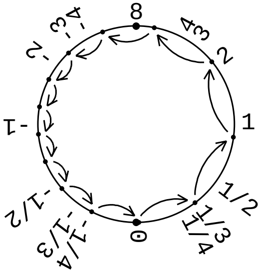
$$M_{a,b,c,d}(x) = \frac{ax+b}{cx+d} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad ad - bc > 0$$



$F_0(x) = x/2$
hyperbolic

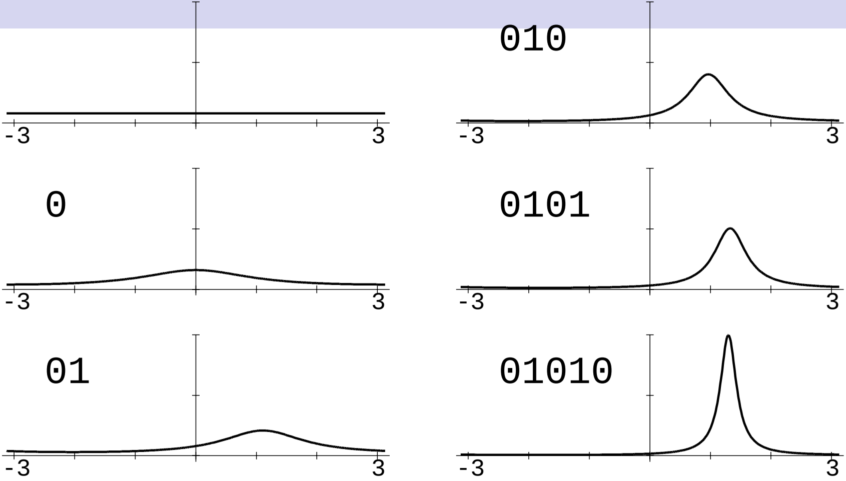


$F_1(x) = x + 1$
parabolic



$F_2(x) = \frac{4x+1}{3-x}$
elliptic

Probability densities: $F_0(x) = \frac{x}{2}$, $F_1(x) = 1 + x$



$$\lim_{n\to\infty} F_{u_{[0,n)}}\ell = \delta_{\Phi(u)}, \;\; u \in A^{\mathbb{N}}$$

The space of Borel probability measures $\mathfrak{M}(X)$

$$\lim_{n\to\infty} \nu_n = \nu \iff \lim_{n\to\infty} \int f d\nu_n = \int f d\nu$$

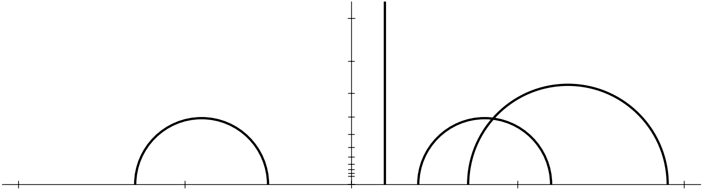
for each continuous $f : X \to \mathbb{R}$.

$F : X \to Y$ determines $F : \mathfrak{M}(X) \to \mathfrak{M}(Y)$

$$(F\nu)(U) = \nu(F^{-1}(U))$$

point measure: $\delta_x(U) = 1 \iff x \in U$

Hyperbolic geometry of the upper half-plane $\mathbb{H} = \{z = x + iy : \; y > 0\}$, $M(z) = (az + b)/(cz + d)$

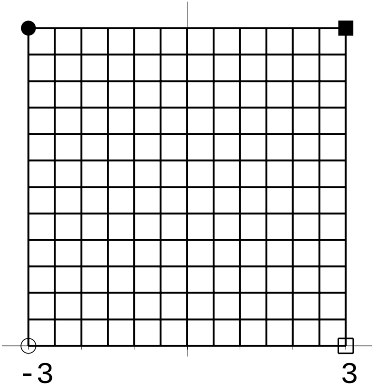


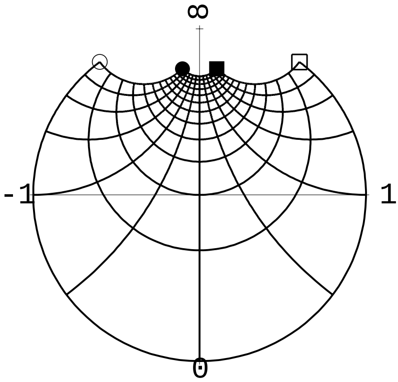
Hyperbolic lines: Eucleidean lines and circles perpendicular to the real axis

Hyperbolic metric: $$ds = \frac{\sqrt{dx^2 + dy^2}}{y}$$

Möbius transformations preserve hyperbolic metric.

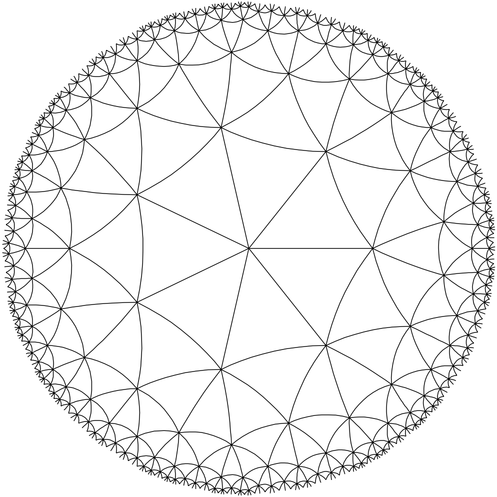
Hyperbolic geometry of the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$



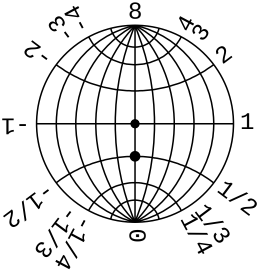


$$\mathbf{d}(z) = \frac{i - z}{i + z}, \mathbf{M} = \mathbf{d} M \mathbf{d}^{-1} : \mathbb{D} \to \mathbb{D}, ds = \frac{\sqrt{dx^2 + dy^2}}{1 - x^2 - y^2}$$

Equilateral triangles in the hyperbolic plane

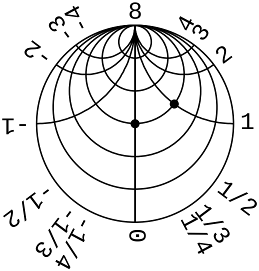


Möbius transformations



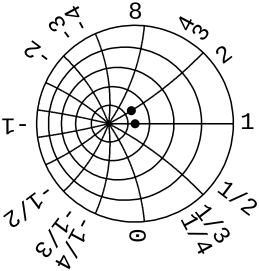
$F_0(x) = x/2$

hyperbolic



$F_1(x) = x + 1$

parabolic

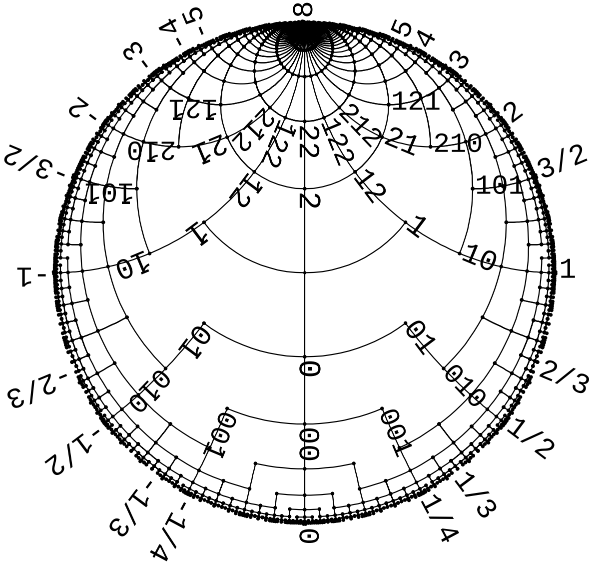


$F_2(x) = \frac{4x+1}{3-x}$

elliptic

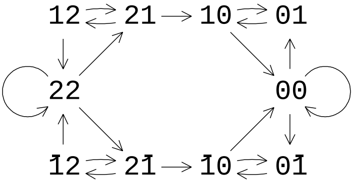
Mean value $\mathrm{E}(M\ell) = \int_{-\pi}^{\pi} (M\ell)(t) e^{it} dt = \mathbf{M}(0)$

Redundant binary system $A = \{\bar{1}, 0, 1, 2\}$

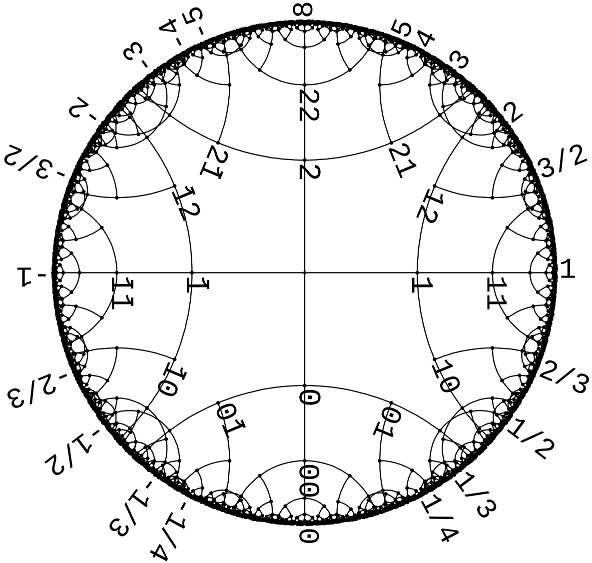


$F_{\bar{1}}(x) = -1 + x$
$F_0(x) = x/2$
$F_1(x) = 1 + x$
$F_2(x) = 2x$

forbidden words:
$02, 20, 1\bar{1}, \bar{1}1, 11, \overline{11},$
$012, 0\bar{1}2, 12\bar{1}, \bar{1}21$



Square system $A = \{\overline{1}, 0, 1, 2\}$

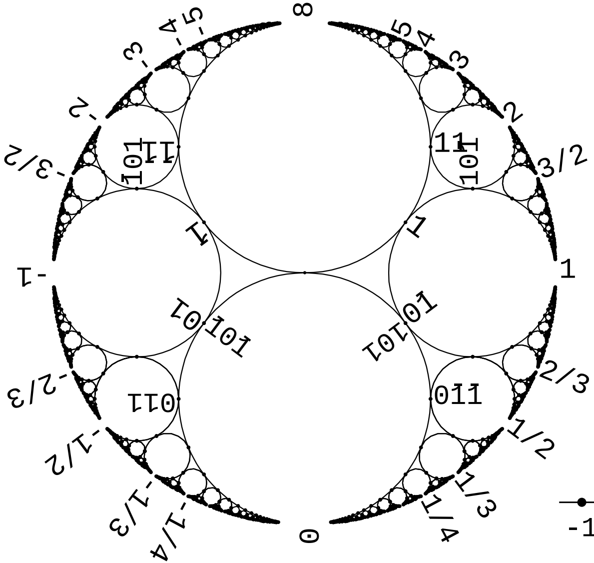


$$F_{\overline{1}}(x) = R_{\frac{3\pi}{2}} F_0 R_{\frac{\pi}{2}}(x)$$
$$F_0(x) = qx$$
$$F_1(x) = R_{\frac{\pi}{2}} F_0 R_{\frac{3\pi}{2}}(x)$$
$$F_2(x) = x/q$$

$$q = (3 - \sqrt{5})/2$$

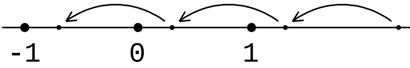
forbidden words:
$02, 20, 1\overline{1}, \overline{1}1,$
$012, 12\overline{1}, 2\overline{1}0, \overline{1}01,$
$210, \overline{1}21, 0\overline{1}2, 10\overline{1}12$

Semi-regular continuous fractions $A = \{\overline{1}, 0, 1\}$

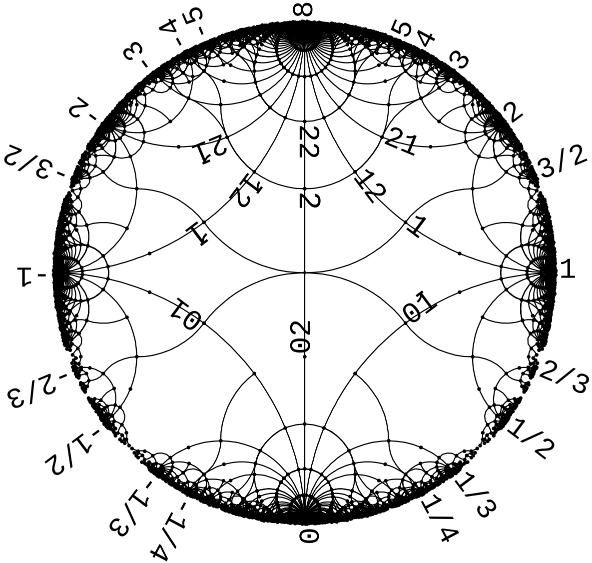


$F_{\overline{1}}(x) = -1 + x$
$F_0(x) = -1/x$
$F_1(x) = 1 + x$

forbidden words:
$00, 1\overline{1}, \overline{1}1, 010, 0\overline{1}0,$



Redundant binary continuous fractions $A = \{\overline{1}, 0, 1, 2\}$



$F_{\overline{1}}(x) = -1 + x$
$F_0(x) = -1/x$
$F_1(x) = 1 + x$
$F_2(x) = 2x$

forbidden words:
$00, 20, 1\overline{1}, \overline{1}1, 11, \overline{1}\overline{1},$
$101, \overline{1}0\overline{1}, 12\overline{1}, \overline{1}21,$
$021, 02\overline{1}, 122\overline{1}, \overline{1}221$

Möbius number system (F, Σ)

$(F_a : \overline{\mathbb{R}} \to \overline{\mathbb{R}})_{a \in A}$: Möbius iterative system

convergence space $\mathbb{X}_F$, $\Phi : \mathbb{X}_F \to \overline{\mathbb{R}}$.

$$\begin{aligned} \mathbb{X}_F &= \{u \in A^{\mathbb{N}} : \ |\lim_{n\to\infty} \mathbf{F}_{u_{[0,n)}}(0)| = 1\} \\ &= \{u \in A^{\mathbb{N}} : \ \lim_{n\to\infty} F_{u_{[0,n)}}(i) \in \overline{\mathbb{R}}\} \\ \Phi(u) &= \lim_{n\to\infty} F_{u_{[0,n)}}(i) \in \overline{\mathbb{R}}, \ u \in \mathbb{X}_F \end{aligned}$$

$\Sigma \subseteq \mathbb{X}_F$ is a sofic subshift,
$\Phi : \Sigma \to \overline{\mathbb{R}}$ is continuous and surjective.

Intervals of contraction and expansion

$$\begin{aligned}
\mathbf{t}(x) &= 2\arctan x,\ \mathbf{t}:\overline{\mathbb{R}} \to \mathbb{T} = [-\pi,\pi), \\
\mathrm{M} &= \mathbf{t}M\mathbf{t}^{-1} : \mathbb{T} \to \mathbb{T} \\
M^{\bullet}(x) &= \mathrm{M}'(\mathbf{t}(x)) = \frac{(ad-bc)(x^2+1)}{(ax+b)^2+(cx+d)^2}
\end{aligned}$$

$$\begin{aligned}
\mathbf{U}_u &= \{x \in \overline{\mathbb{R}} :\ F_u^{\bullet}(x) < 1\}, \\
\mathbf{V}_u &= \{x \in \overline{\mathbb{R}} :\ (F_u^{-1})^{\bullet}(x) > 1\} \\
F_u(\mathbf{U}_u) &= \mathbf{V}_u,\ u \in A^{+}
\end{aligned}$$

Theorem. $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ Möbius iterative system.
1. If $\{\overline{\mathbf{V}}_u : \ u \in A^+\}$ do not cover $\overline{\mathbb{R}}$, then $\Phi(\mathbb{X}_F) \neq \overline{\mathbb{R}}$.
2. If $\{\mathbf{V}_u : \ u \in A^+\}$ cover $\overline{\mathbb{R}}$, then $\Phi(\mathbb{X}_F) = \overline{\mathbb{R}}$ and there exists a subshift $\Sigma \subseteq \mathbb{X}_F$ such that $\Phi : \Sigma \to \overline{\mathbb{R}}$ is surjective and continuous.

$$x \in \mathbf{V}_{u_0}, \ F_{u_0}^{-1}(x) \in \mathbf{V}_{u_1}, \ F_{u_{[0,1]}}^{-1}(x) \in \mathbf{V}_{u_2}, \ldots$$

Convergence theorem

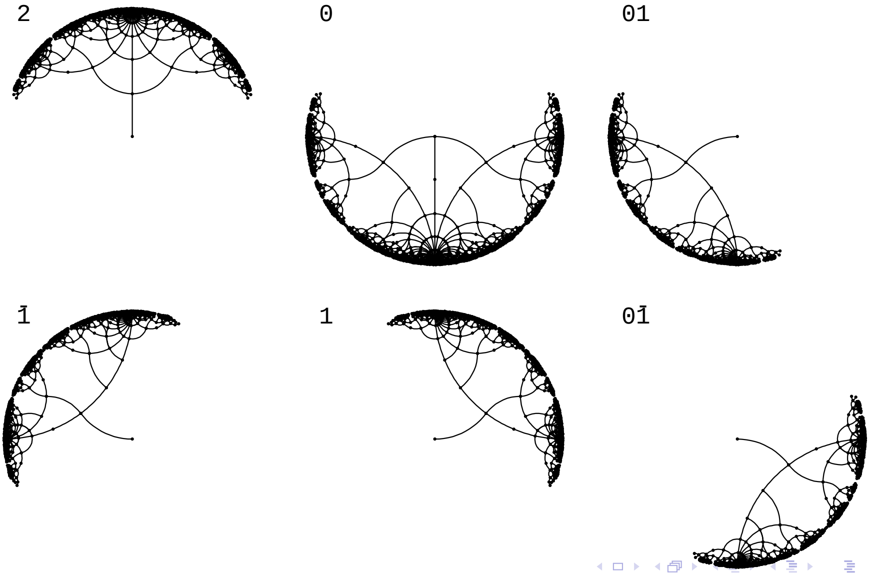
Theorem. $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ Möbius iterative system, $G = (V, E, s, t, h)$ labelled graph, $(I_q)_{q \in V}$ closed intervals such that for each $e \in E$

$$I_{t(e)} \subseteq \overline{\mathbf{U}}_{h(e)}, \; F_{h(e)}(I_{t(e)}) \subseteq I_{s(e)}$$

Then $\Sigma_G \subseteq \mathbb{X}_F$ and $\Phi : \Sigma_G \to \overline{\mathbb{R}}$ is continuous.

$$\xrightarrow{a} p \xrightarrow{b} q \implies F_b(\mathbf{U}_b) \subseteq \mathbf{U}_a$$

Cylinders of a redundant system



$K : V \rightarrow E$ is a selector for $G = (V, E, s, t, h)$, if $s(K(q)) = q$, i.e., $q \stackrel{K(q)}{\rightarrow}$

$\mathcal{O}(q) \subseteq E^{\mathbb{N}}$: paths with source $q \in V$.

A selector K determines a (pre)periodic path $k^q \in \mathcal{O}(q)$: $k_0^q = q$, $k_{i+1}^q = K(t(k_i^q))$

Cylinder of $q \in V$: $[q]_\Phi = \Phi(h(\mathcal{O}_q)) \subseteq \overline{\mathbb{R}}$

Surjectivity theorem

Theorem. If $F : A^+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a Möbius iterative system, and $G = (V, E, s, t, h)$ graph such that $\Sigma_G \subseteq \mathbb{X}_F$ and $\Phi_F : \Sigma_G \to \overline{\mathbb{R}}$ is continuous, then there exist selectors L, R with

$$[p]_\Phi \subseteq J_p := [\Phi_F(h(l^p)), \Phi_F(h(r^p))].$$

If J_p cover $\overline{\mathbb{R}}$ and if

$$J_p \subseteq \bigcup \{F_{h(e)}(J_q) : \ p \xrightarrow{e} q\}.$$

then $\Phi(\Sigma_G) = \overline{\mathbb{R}}$ and $[p]_\Phi = J_p$ for $p \in V$.

Aritmetical algorithms

$(F_a : \overline{\mathbb{R}} \to \overline{\mathbb{R}})_{a \in A}$ Möbius number system with integer coefficients, $\Sigma \subseteq A^{\mathbb{N}}$, sofic subshift.
$\Phi : \Sigma \to \overline{\mathbb{R}}$ surjective, continuous and redundant
$\mathcal{A} = (Q, \delta, q_0, q_1)$ finite automaton for $\mathcal{L}(\Sigma)$
$\delta : Q \times A \to Q$, $q_0, q_1 \in Q$

Endpoints of cylinders $J_q = [q]_\Phi = \Phi_F(h(\mathcal{O}_q))$ are quadratic numbers.

$$\Phi([u]) = F_u(J_{\delta(q_0,u)}),\ u \in \mathcal{L}(\Sigma)$$

function sum(u,v:input words; w:output word)

$p_u := q_0$; $p_v := q_0$; $p_w := q_0$;
$M_u := \mathrm{Id}$; $M_v := \mathrm{Id}$; $M_w := \mathrm{Id}$; $I_u := \overline{\mathbb{R}}$; $I_v := \overline{\mathbb{R}}$;
repeat
    $I_w := M_w^{-1}(I_u + I_v)$;
    If $(\exists q, a)(p_w \xrightarrow{a} q \;\&\; I_w \subseteq F_a(J_q))$ then begin
        $p_w := q$; $M_w := M_w \cdot F_a$;
        write a to w; end;
    else begin
        If $|I_u| > |I_v|$ then begin
            read a from u;
            $M_u := M_u \cdot F_a$; $p_u := \delta(p_u, a)$; $I_u := M_u(J_{p_u})$; end;
        else begin
            read a from v;
            $M_v := M_v \cdot F_a$; $p_v := \delta(p_v, a)$; $I_v := M_v(J_{p_v})$; end;
        end;
    end;