Harmonic properties of some automatics flows

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I.1 q-automatic sequence

Classical : A sequence u in a set X is said to be q-automatic if the set $G(u)$ of subsequences

$$n \mapsto u(q^k n + r), \quad 0 \leq t < q^k \quad (k \in \mathbf{N}$$

is finite.
Let $G(u) := \{g_0, \dots, g_{m-1}\}$ with $g_0 = u$ and define $\gamma : n \rightarrow X^m$ by

$$\gamma(n) = \begin{bmatrix} g_0(n) \\ \vdots \\ g_{m-1}(n) \end{bmatrix} .$$

For any q-digits $j = 0, 1, \ldots, q-1$, there exists a matrix A_j (called instruction map) with 0-1 entries defined by the relation

$$\gamma(qn+j) = A_j \gamma(n) \quad (n \in \mathbf{N})$$

Notice :

• each row contains only one 1,
• these 1 are symbolic (playing the rôle of a selection operator).

The sequence u is generated by the *standard* q-automaton defined by :

- space of states : $\gamma(\mathbf{N})$,
- initial state : $\gamma(0)$,
- instructions : $A_j \quad (0 \leq j < q)$,

and the sequence u is obtained from the output map :

$$\gamma(n) \mapsto g_0(n) \qquad \text{(first projection)}.$$

I.2 Linear representation

For a given automatic sequence u, there are many automata that generate u, but there always exists a linear model, *i.e.,*

- the space of states is a subset of a linear space E ;
- each instruction can be extended to a linear endomorphism (say A_j) of E.

For example, if u is real or complex valued, the above *standard* automaton furnished a *standard linear* model that generates u with a minimal number of states.
$\rightarrow$ keep in mind that $A_j(\gamma)(n) = \gamma(qn + j)$
and

$$A_i A_j(\gamma)(n) = \gamma(q^2 n + qj + i)$$

A concrete example :

Let u be the periodic sequence of period 3 defined by

$$u(0) = 1,\ u(1) = u(2) = -1.$$

Classically, u is 2-automatic. If we introduce the translation map $T : n \mapsto n + 1$, one has in fact

$$G(u) := \{g_0 = u,\ g_1 = u \circ T,\ g_1 = u \circ T^2\}$$

with instructions

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and space of states $\gamma(\mathbf{N}) = \left\{ \begin{bmatrix} +1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ +1 \end{bmatrix}, \begin{bmatrix} -1 \\ +1 \\ -1 \end{bmatrix} \right\}$

II. SUMMATION FORMULA

I.1 formal summation

We use the standard model (to fix the ideas)

For $n = \sum_{r=0}^{t} e_r(n) q^r$ (standard) :

$$\gamma(n) = A_{e_0(n)} \dots A_{e_t(n)} \gamma(0) .$$

And we are interested in the formal sum

$$\boxed{\Gamma(N; z) := \sum_{n<N} \gamma(n).z^n}$$

which is a comfortable manner :=)) to write the object

$$(\gamma(0), \gamma(1), \dots, \gamma(N-1), \emptyset, \emptyset, \emptyset, \dots)$$

Now we introduce the formal sum

$$A[z] := A_0 + zA_1 + \cdots + z^{q-1}A_{q-1} (= (A_0, \ldots, A_{q-1}, \wedge, \wedge, \wedge \ldots)$$

($\wedge$ for *cancel operator*) which acts on $\Gamma(N; z)$ by distributivity of the local actions

$$z^k A_k(z^m \gamma(m)) = z^{k+m} \gamma(qm + k)$$

so that

$$\boxed{\Gamma(qN, z) = A(z)\Gamma(N, z^q)}$$

and ("block" summation) :

$$\boxed{\Gamma(q^m, z) = A(z)A(z^q) \ldots A(z^{q^{m-1}})\Gamma(1; z^{q^m}) , \quad \Gamma(1; z^{q^m}) = \gamma(0)}$$

Formal sum (continued)

Set

$$\Pi_m(z) = A(z)A(z^q)\dots A(z^{q^{m-1}})$$

For $N = \sum_{0\le r\le K} e_r(N)q^r \quad (e_k(N) \neq 0)$ and for $0 \le m \le K$, define the m-tail

$$t_m = \sum_{m\le r\le K} e_r(N)q^r \quad \text{and} \quad t_{K+1} = 0.$$

From above we derive the formal summation

$$\boxed{\Gamma(N,z) = \sum_{m\ge 0} z^{t_{m+1}}\Pi_m(z)e_m(1,N,z^{q^m})\gamma\left(\left\lfloor \frac{N}{q^{m+1}}\right\rfloor\right)}$$

with $e_m(A,N,\cdot) = \wedge$ if $e_m(N) = 0$ and

$$e_m(A,N,z^{q^m}) = \sum_{j<e_m(N)} z^{jq^m}A_j \quad \text{otherwise.}$$

Interest ?

(1) If X is a compact space, in order to study u from a statistical and harmonical point of view, it is classical to replace $u(n)$ by $f(u(n))$ where f is a continuous map.

(2) If X is a compact metrizable group, it is useful to introduce irreducible representations ρ of X and then to replace $u(n)$ by orthogonal matrices $\rho(u(n))$.

In both case z figure a complex number of modulus 1.

With a linear representation of the automaton, the instructions turn to be matrices, the operator $A(z)$ can be viewed as a matrix and then, formal sums become summation in a suitable linear space.

The usual goal is to estimate these sums (used in the ergodic machinery). A way to attack this problem is to compute the quadratic operator norm of the matrix $A(z)$:

$$||A(\zeta)||_2 = \sqrt{\max\{\text{eigenvalues of } A(\zeta)^* A(\zeta)\}}.$$

III.1 Some generalisation

Going back to the output formula

$$\gamma(n) = A_{e_0(n)} \dots A_{e_t(n)} \gamma(0)$$

one can decide to change the automaton on line, at each step, taking care that the corresponding output

$$\gamma(n) = A^0_{e_0(n)} \dots A^{(t)}_{e_t(n)} \gamma(0)$$

is meaning full. Summation formula remains unchanged and estimation by quadratic norm can be a fruitful tool.

A nice example (for me, but for you ?)

Choose the following linear realizations of Thue-Morse sequence

$$S^{(0)} = \left\{ \begin{bmatrix} +1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ +1 \end{bmatrix} \right\}, \quad A_0^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1^{(0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

and Rudin-Shapiro sequence

$$S^{(1)} = \left\{ \begin{bmatrix} +1 \\ +1 \end{bmatrix}, \begin{bmatrix} +1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ +1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}, \quad A_0^{(1)} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_1^{(1)} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

Fine ! the matrices $A_0^{(0)}$ and $A_1^{(0)}$ also act on $S^{(1)}$, leading to the following 2-*stack-automatic* sequences :

Construction

Choose $(\varepsilon_n)_n \in \{0,1\}^{\mathbf{N}}$ and define $\gamma^{(\varepsilon)} : N \to \{-1,+1\}^2$ by

$$\gamma^{(\varepsilon)}(n) = A_{e_0(n)}^{\varepsilon_0} \dots A_{e_t(n)}^{(\varepsilon_t)} \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

We have for $|z| = 1$ (easy)

$$||A^{(0)}(z)||_2 \leq 2 \text{ and } ||A^{(1)}(z)|| \leq \sqrt{2}\,.$$

Therefore

$$||\Gamma(q^m, z)||_2 \leq c.2^{\frac{1}{2}(\varepsilon_0+\dots+\varepsilon_{m-1})}$$

and

$$\boxed{||\Gamma(N,z)||_2 \leq C.\sum_{r=0}^{\infty} e_r(N) 2^{\frac{1}{2}(\varepsilon_0+\dots+\varepsilon_{t-1})}}\,;$$

finally we are able to infer many interesting properties of the dynamical system built from the sequences $\gamma^{(\epsilon)}$.

IV. CHAINED SEQUENCES

IV.1 Main definitions

X is a compact metrizable group denoted by G.

Definition. A map $f : \{0, 1, \ldots, q-1\}^* \to G$ is (q)-chained if f(empty word) $= 1_G$ and for all digits a, b and all digital words w one has

$$f(abw) = f(ab)f(b)^{-1}f(bw).$$

Consequently

$$f(a_1a_2\cdots a_sw) = f(a_1a_2)f(a_2)^{-1}\ldots f(a_{s-1}a_s)f(a_s)^{-1}f(a_sw)\,.$$

The chained map is said to be left regular if $f(0w) = f(w)$ for any digital word w.

Definition. a sequence $u : N \to G$ is chained if there exists a left regular chained map f such that

$$u_n = f(e_t(n) \cdots e_0(n)) \quad (n = \sum_{r=0}^{t} e_r(n) q^r).$$

Typical examples :

– Completely q-multiplicative sequences are chained, in particular the sum-of-digits function ;

– Rudin-Shapiro sequence is chained (it is a 2-bloc map).

Notice that the underlying chained maps f are right regular i.e., $f(w0) = f(w)$ for all digital words.

Transition matrix T

$$^{a}T_b = f(ab)f(b)^{-1} .$$

For any irreducible representation ρ of G the matrices $\rho T = (\rho(^{a}T_b))_{a,b}$ verifies :

$$\boxed{\sqrt{q} \leq ||\rho T||_2 \leq q}.$$

The chaines sequence is called :

- contractive if $||\rho T||_2 < q$,
- Hadamard if $||\rho T||_2 = \sqrt{q}$,

for all irreducible non trivial representations of G.

The case $||\rho T|| = 2$ is typically represented by

$$^{i}\rho T_j = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & ^{i}B_j & \\ 0 & & & \end{bmatrix}$$

Summation

For all chained sequences in G and all q-multiplicative maps $\varphi : \mathbf{N} \to \mathbf{U}$ one has (J-P. Allouche, P.L. in AA91)

$$|| \sum_{n<N} \varphi(n+k)\rho.u(n+k)|| \leq 1 + c(\rho.u)2^{\frac{2}{3}-\alpha(\rho,T)}q^{1/2}(1+q^{1/2})N^{\alpha(\rho,f)}$$

where $\alpha(\rho,f) = \frac{\log ||\rho T||}{\log q}$ and $c(\rho,u) = \frac{q-1}{q^{\alpha(\rho,u)}-1}$.

Interesting consequences

Let K_u be the obit closure of u according to the shift transformation S on $G^{\mathbf{N}}$. Let

$$K(u) = (S, K_u)$$

the flow obtained by restriction of S onto K_u.

(1) For any chained sequence, the flow $K(u)$ is minimal.

(2) Periodic chained sequence are of the form

$$u_n = g^n g^{-e_0(n)} u(e_0(n))$$

where g is of order $q - 1$.

(3) If the transition matrix is contractive, $K(u)$ is uniquely ergodic, topologically conjugate to a “standard skew product”.

Skew product

Let $\Phi_u : \mathbf{Z}_q \to G$ denote the cocycle associated to u, defined by

$$\Phi_u(x) = \lim_{n \to x, n \in \mathbf{N}} u_n^{-1} u_{n+1} \quad (x \neq -1)$$

– If u is contractive the $K(u)$ is metrically conjugate to the skew product τ_u defined on $\mathbf{Z}_q \times G$ by

$$\tau_u(x, g) = (x + 1, g\Phi_u(\alpha)).$$

– If u is not contractive and not periodic, there exists a G-valued sequence v, completely q-multiplicative, and a periodic sequence p of period q such that the flow $K(u)$ is strictly ergodic, metrically conjugate to $K(v)$.

– The corresponding skew product τ_u is more complicated to study. Notions of **essential topological values** of u and **index of periodicity** allow to identify the cases where the flow is metrically conjugate to the skew product τ_u.

– The spectral type of the skew product can be described.

If u is Hadamard, then the spectral type of the skew product τ_u on the orthocomplement of the odmeter $(\mathbf{Z}_q, x \mapsto x+1)$ is Lebesgue.

• In the abelian case, the stucture of Hadamard sequences is simple.

$\rightarrow$ The typical example (but not the only one) was given by Martine Queffélec (AIF87) with q prime and the transition matrix

$$^{i}T_j = e^{2i\pi ij/q} .$$

V. ILLUSTRATION

For $q = 2$, let $s(\cdot)$ be the sum-of-digits function, $s_{11}(\cdot)$ the counting map of the pattern "11" in binary expansions and let θ be a complex number of modulus 1.
The sequence $n \mapsto \theta^{e_0(n)}(-1)^{s(n)} i^{s_{11}(n)}$ is chained in base 2.
$G = [\theta]\mathbf{U}_4$ and the transition matrix is $\begin{bmatrix} 1 & 1 \\ -1 & -i \end{bmatrix}$;
The group of essential values is $\mathbf{U}_4$.

We collect from the above machinery :

– If $\theta \in \mathbf{U}_4$ then u is contractive, and $K(u)$ is metrically conjugate to the skew-product.

– If $\theta \notin \mathbf{U}_4$, the sequence u is not contractive but the flow $K(u)$ is strictly ergodic, topologically conjugate to the flot $K(v)$ with $v_n = (-1)^{s(n)} i^{s_{11}(n)}$. This last sequence is chained, contractive and the sequence $n \mapsto v_n^2$ is Hadamard.

VI. GENERALISATION

Chained sequences are, in fact, particular automatic sequences, and the previous construction with q-stack-automata works well.

Here we chose a sequence of transition matrix to define the sequence :

$$U_n = {}^{e_t(n)}T^{(t)}_{e_{t-1}(n)} \cdots {}^{e_1(n)}T^{(1)}_{e_0(n)} U_{e_0(n)}$$

The simplest case correspond to contractive transition matrices occuring, at least, with a positive density.

Thanks for your attention !