Crystallographic number systems

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Prag - May, 2008

Supported by FWF, Project S9610.

Introduction.

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Results: correspondances with canonical number systems.

Crystallographic groups.

$\Gamma \leq \mathrm{Isom}(\mathbb{R}^n)$ is a *crystallographic group* if

$$\Gamma \simeq \mathbb{Z}^n \ltimes \{id, r_2, \ldots, r_d\},$$

where $r_2, \ldots, r_d$ are isometries of finite order.

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$n = 2$: 17 crystallographic groups. Example :

$$a(x, y) = (x+1, y), \quad b(x, y) = (x, y+1), \quad c(x, y) = (-x, -y).$$

A $p2$-group is isomorphic to the group generated by a, b, c.

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and

$$g(\mathcal{T}) = \bigcup_{\delta\in\mathcal{D}} \delta(\mathcal{T}) \tag{2}$$

(*replication property*).

Example of a p_2-crystile.

$$g(x, y) = \begin{pmatrix} 0 & -3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

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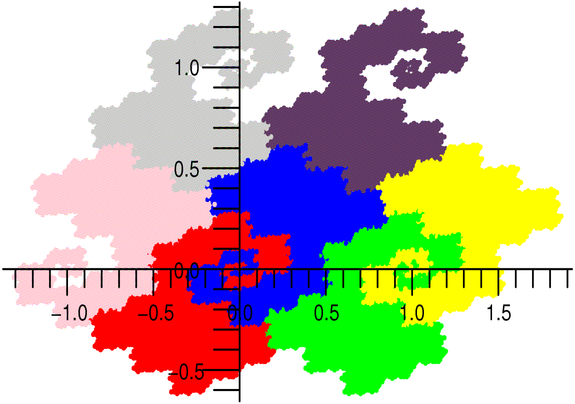

Figure: $\mathcal{T}$ defined by $g(\mathcal{T}) = \mathcal{T} \cup a(\mathcal{T}) \cup c(\mathcal{T})$ and its neighbors.

Associated dynamical system.

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Iterating Φ, one gets

$$\begin{array}{rcl} \gamma & = & g\gamma_1 g^{-1}\delta_0 \\ & = & g\ g\gamma_2 g^{-1}\delta_1 g^{-1}\ \delta_0 \\ & = & \dots \\ & = & g^m\Phi^m(\gamma)g^{-1}\delta_{m-1}\dots g^{-1}\delta_1 g^{-1}\delta_0 \end{array}$$

with digits $\delta_0, \dots, \delta_{m-1} \in \mathcal{D}$.

Crystallographic number system.

Definition. $(\Gamma, g, \mathcal{D})$ is a *crystallographic number system* if for every $\gamma \in \Gamma$, $\Phi^m(\gamma) = id$ for some $m \in \mathbb{N}$. One then writes $\gamma = (id^\infty, \delta_{m-1}, \ldots, \delta_0)_g$ or just $(\delta_{m-1}, \ldots, \delta_0)_g$.

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The digit set has the form

$$\mathcal{D} = \left\{ \mathbf{x} \mapsto \mathbf{x} + \begin{pmatrix} p_i \\ q_i \end{pmatrix} \; ; \; 1 \leq i \leq d \right\}.$$

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$(\Gamma, g, \mathcal{D})$ is a *crystallographic number system* iff $\left(\mathrm{M}, \mathcal{N} := \left\{ \begin{pmatrix} p_i \\ q_i \end{pmatrix} ; 1 \leq i \leq d \right\}\right)$ is a *number system*.

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If

$$\gamma = (id^{\infty}, \delta_{m-1}, \dots, \delta_0)_g,$$

then

$$\gamma\gamma_0 = g^m \gamma' g^{-m} (\delta'_{m-1}, \dots, \delta'_0)_g$$

where

$$\gamma_0 \xrightarrow{\delta_0|\delta'_0} \gamma_1 \xrightarrow{\delta_1|\delta'_1} \dots \xrightarrow{\delta_{m-1}|\delta'_{m-1}} \gamma'.$$

Example of $p2$-crystem ($\Gamma =< a, b, c >$).

$$g(x,y) = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}.$$

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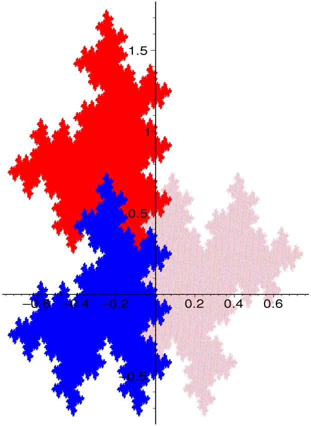

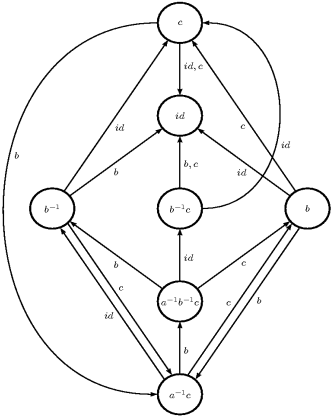

Figure: $\mathcal{T} : g(\mathcal{T}) = \mathcal{T} \cup b(\mathcal{T}) \cup c(\mathcal{T})$ and counting subautomaton.

Characterization by a subautomaton.

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$$\gamma \xrightarrow{id|\delta_0} \gamma_1 \xrightarrow{id|\delta_1} \dots \xrightarrow{id|\delta_{m-1}} id \tag{3}$$

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in the counting automaton.

Suppose Property (3) is fulfilled by the states of a stable subautomaton that generates Γ. Then $(\Gamma, g, \mathcal{D})$ is a crystem.

Example of $p2$-non-crystem ($\Gamma =< a, b, c >$).

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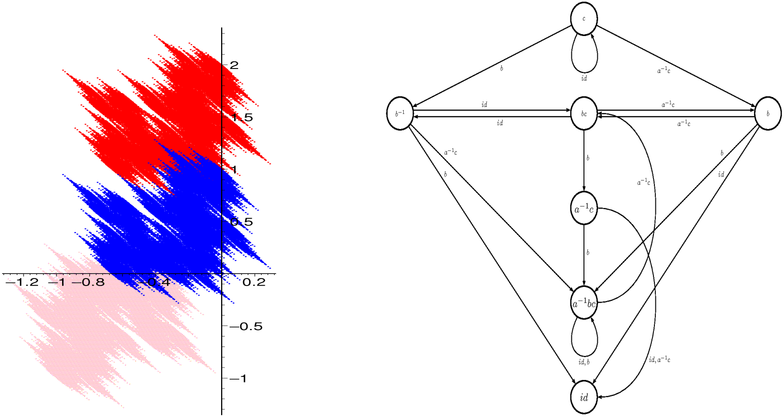

Figure: $\mathcal{T} : g(\mathcal{T}) = \mathcal{T} \cup b(\mathcal{T}) \cup a^{-1}c(\mathcal{T})$ and counting subautomaton.

Example of $p3$-crystem

$\Gamma =< a, b, r >$, $b(x, y) = (x + \frac{1}{2}, y + \frac{\sqrt{3}}{2}), r = \mathrm{rot}(0; \frac{2\pi}{3}))$.

$$g(x, y) = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

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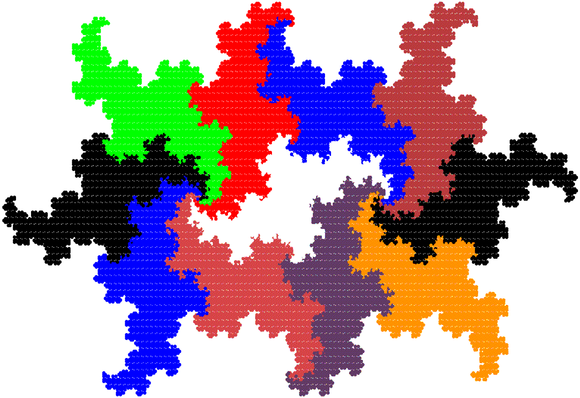

Figure: $\mathcal{T}\ :\ g(\mathcal{T}) = \mathcal{T} \cup ac^2(\mathcal{T}) \cup bc^2(\mathcal{T})$ and its neighbors.

Complete sets of right coset representatives.

Let $(\Gamma, g, \mathcal{D})$ with Γ crystallographic group,

$$g(x) = \mathrm{M}x + t$$

$$\mathcal{D} = \{id, \mathrm{D}_i(x) + d_i\}.$$

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Then
$\mathcal{D}$ is a complete set of right coset representatives of $\Gamma / g \Gamma g^{-1}$
iff

$$\mathcal{N} := \{\mathrm{M}\mathrm{D}_i^{-1}\mathrm{M}^{-1} d_i + (\mathrm{I}_n - \mathrm{M}\mathrm{D}_i^{-1}\mathrm{M}^{-1}) t\}$$

is a complete set of coset representatives of $\mathbb{Z}^n / \mathrm{M}\mathbb{Z}^n$.

A class of $p2$-crystems.

$$\Gamma = \{a^p b^q c^r; p, q \in \mathbb{Z}, r = 0, 1\},$$

$$g(x,y) = \underbrace{\begin{pmatrix} \alpha & \beta \\ \epsilon & \delta \end{pmatrix}}_{\mathrm{M} \in \mathbb{Z}^2 \times \mathbb{Z}^2} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{B-1}{2} \\ 0 \end{pmatrix},\ |\mathrm{det}(\mathrm{M})| = B \geq 2.$$

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Then $\mathcal{D}$ is a complete set of coset rep. iff $\mathcal{N} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} B-2 \\ 0 \end{pmatrix}, \begin{pmatrix} B-1 \\ 0 \end{pmatrix} \right\}$ is a complete set of coset rep. of $\mathbb{Z}^2/\mathrm{M}\mathbb{Z}^2$ (iff $\epsilon = \pm 1$).

A class of $p2$-crystems : g-basis and M-basis.

Suppose $\gamma = a^p b^q = (id^\infty, \delta_{m-1}, \ldots, \delta_0)_g$ with $\delta_0, \ldots, \delta_{m-1} \in \mathcal{D}$.

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Then $\begin{pmatrix} p \\ q \end{pmatrix} = (0^\infty, \delta'_p, \ldots, \delta'_0)_{\mathrm{M}}$ where the digits $\delta'_1, \ldots, \delta'_p \in \mathcal{N}$ are obtained via the two-states automaton:

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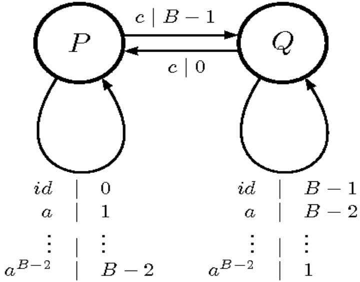

Figure: Exchange automaton $g - \mathrm{M}$.

A class of $p2$-crystems : equivalence to canonical number systems.

Hence, $(g, \mathcal{D})$ is a crystem iff $(\mathrm{M}, \mathcal{N})$ is a canonical number system iff $-1 \leq -\mathrm{Tr}(\mathrm{M}) \leq B \geq 2$.

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If $\mathcal{T}$ satisfies $g(\mathcal{T}) = \bigcup_{\delta \in \mathcal{D}} \delta(\mathcal{T})$, then

$$\mathcal{T}^{latt} := \mathcal{T} \cup -\mathcal{T} + (\mathrm{M} - \mathrm{I}_2)^{-1} \begin{pmatrix} \frac{B-1}{2} \\ 0 \end{pmatrix}$$

satisfies

$$\mathrm{M}\mathcal{T}^{latt} = \mathcal{T}^{latt} \cup \mathcal{T}^{latt} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cup \ldots \cup \mathcal{T}^{latt} + \begin{pmatrix} B-1 \\ 0 \end{pmatrix}$$

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$\mathcal{T}$ is disk-like implies $\mathcal{T}^{latt}$ is disk-like.

Conclusion.

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Conjecture : there is still an equivalence between $(g, \mathcal{D})$ *is a crystem* and *the corresponding* $(\mathrm{M}, \mathcal{N})$ *is a number system*.