Eigenvalues of Fibonacci stochastic adding machine

A. Messaoudi, D. Smania

Let $N \in \mathbb{N}$

$$N = \sum_{i=0}^{k} \varepsilon_i(N)2^i = \varepsilon_k(N) \ldots \varepsilon_0(N)$$

where

$\varepsilon_i(N) = 0$ or 1 for all i.

It is known that there exists an algorithm that computes the digits of $N + 1$.

Ex:

$$1011 + 1 = 1100$$

$$
\begin{array}{r}
{}^{0}1^{1}0^{1}11 \\
1 \\
1100
\end{array}
$$

This algorithm can be described by by the following manner:

$c_{-1}(N+1) = 1,$

$$\varepsilon_i(N+1) = \varepsilon_i(N) + c_{i-1}(N+1) mod(2)$$

$$c_i(N+1) = [\frac{\varepsilon_i(N) + c_{i-1}(N+1)}{2}].$$

What happens if the machine dos not work.

P.R. Killeen and T.J. Taylor [KT] consider fallible adding machine by the following:

They consider the algorithm:

$$\varepsilon_i(N+1) = \varepsilon_i(N) + e_i(N)c_{i-1}(N+1) mod(2)$$

$$c_i(N+1) = [\frac{\varepsilon_i(N) + e_i(N)c_{i-1}(N+1)}{2}],$$

where

$$e_i(N) = 1 \text{ with probability } p$$

$$e_i(N) = 0 \text{ with probability } 1-p$$

is an independent, identically distributed family of random variables.

Hence the transition graph is:

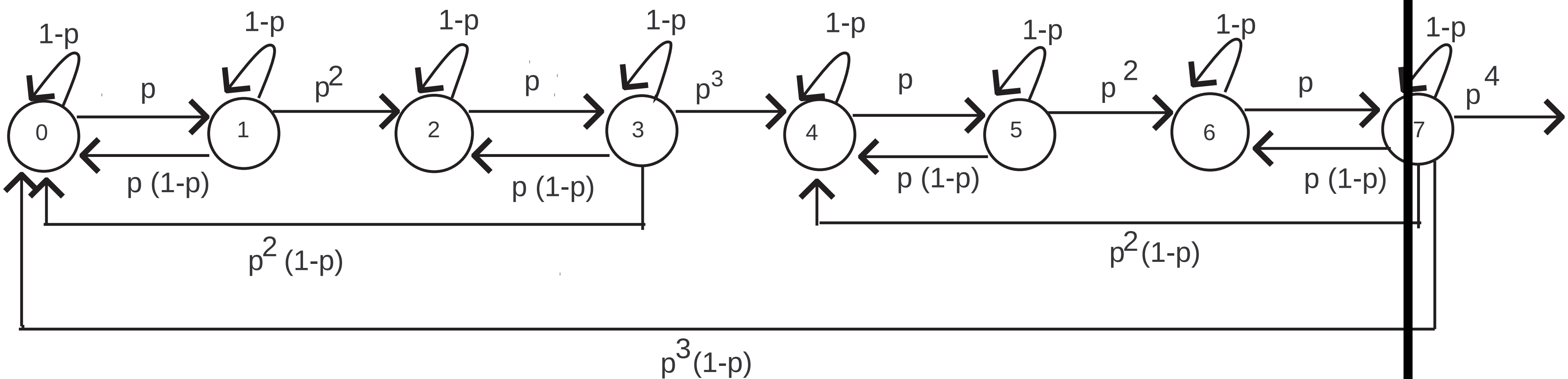

Figure 1: Transition graph of adding machine in base 2

The transition operator P associated to the transition graph is:

$P =$

$$
\begin{pmatrix}
1-p & p & 0 & 0 & 0 & 0 & 0 & 0 \ldots \\
p(1-p) & 1-p & p^2 & 0 & 0 & 0 & 0 & 0 \ldots \\
0 & 0 & 1-p & p & 0 & 0 & 0 & 0 \ldots \\
p^2(1-p) & 0 & p(1-p) & 1-p & p^3 & 0 & 0 & 0 \ldots \\
0 & 0 & 0 & 0 & 1-p & p & 0 & 0 \ldots \\
0 & 0 & 0 & 0 & p(1-p) & 1-p & p^2 & 0 \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 1-p & p \ldots \\
p^3(1-p) & 0 & 0 & 0 & p^2(1-p) & 0 & p(1-p) & 1-p \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
$$

In [KT], P.R. Killen and J. Taylor study the spectrum of the operator P. They prove that the spectrum of P in $l^\infty(\mathbb{N})$ is connected to the Julia set of f where $f : \mathbb{C} \mapsto \mathbb{C}$ defined by: $f(z) = (z - (1-p))^2/p^2$. In particular the set of eigenvalues E satisfies

$$E = \{z \in \mathbb{C}, f^n(z) \text{ bounded } \}$$

1 Fibonacci base

$$F_0 = 1, F_1 = 2,\ F_n = F_{n-1} + F_{n-2}\ \forall n \geq 0.$$

$$N = \sum_{i=0}^{k} \varepsilon_i(N) F_i = \varepsilon_k(N) \ldots \varepsilon_0(N)$$

where

$$\varepsilon_i = 0, 1, \ \varepsilon_i \varepsilon_{i+1} \neq 11, \ \forall 0 \leq i \leq k(N) - 1$$

$$1, 2, 3, 5, 8, 13, 21, 34, 55, ...$$

$$51 = 34 + 13 + 3 + 1 = F_7 + F_5 + F_2 + F_0 = 10100101$$

It is known (see Frougny) that the addition of 1 in base $(F_n)_{n \geq 0}$ is given by a finite transductor.

How to construct this transductor

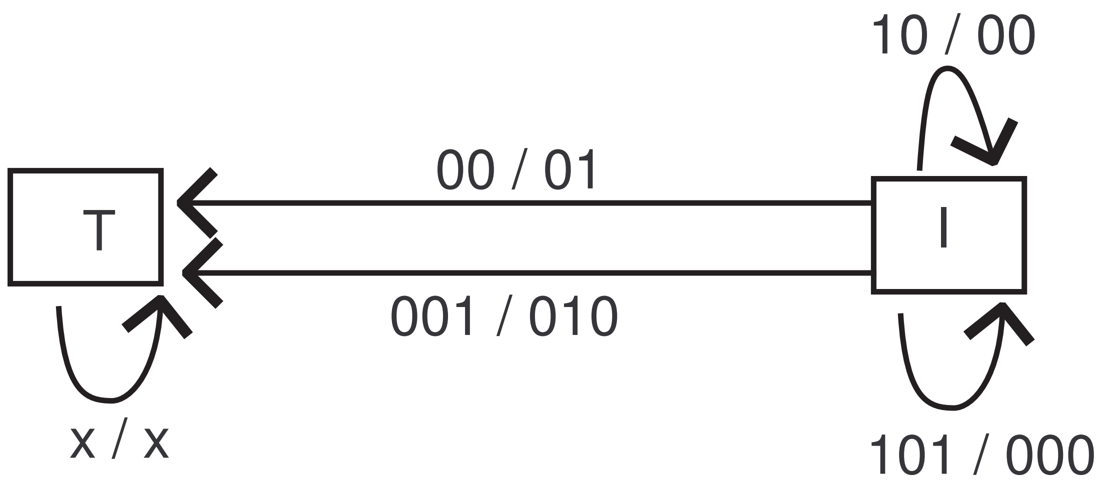

Figure 2: Transductor of Fibonacci adding machine

Ex:

$$N = 10010$$

$$
\begin{array}{r}
10010 \\
1 \\
10011 = 10100
\end{array}
$$

If $N = 10010$ then

$$N = (I, 10/00, I)\ (I, 00/01, T)\ (T, 1, T).$$

Hence

$$N + 1 = 10100.$$

Now we define the stochastic adding machine by the following manner

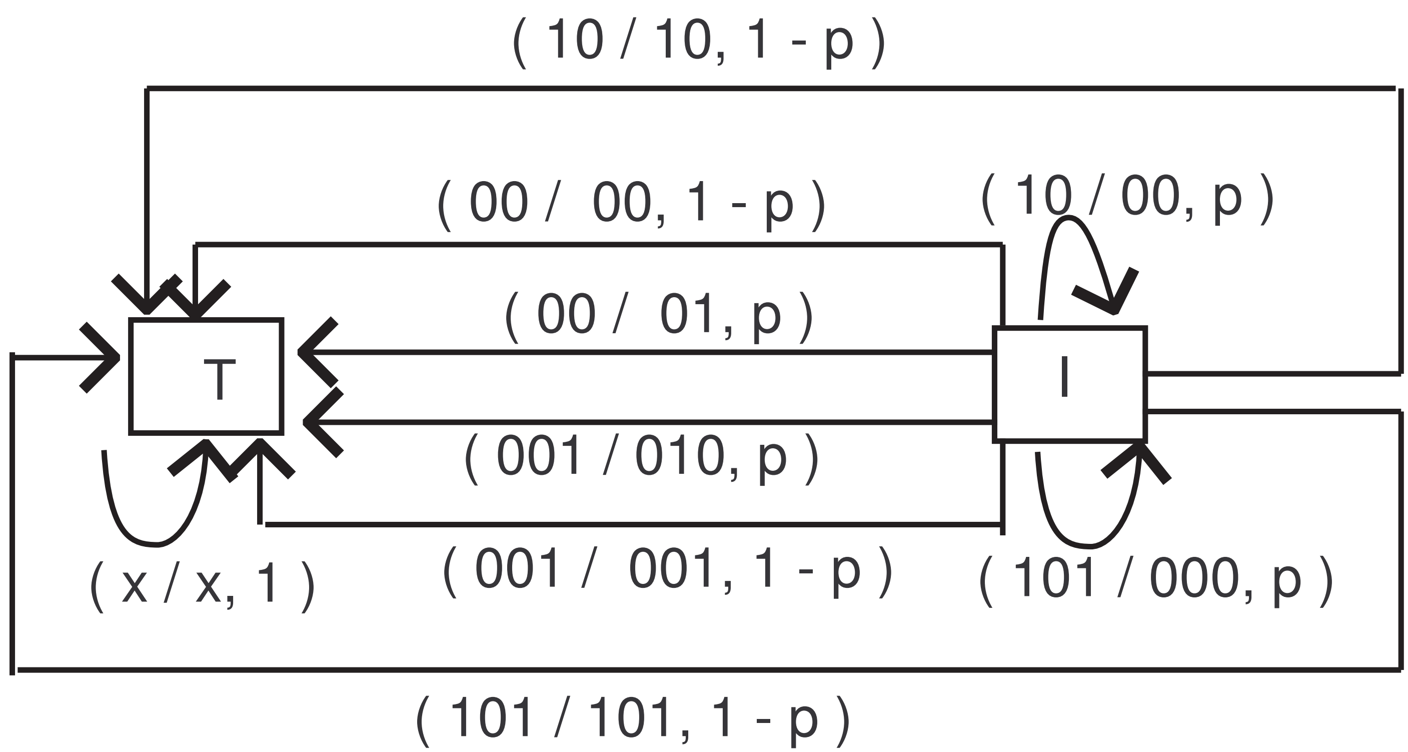

Figure 3: Transductor of Fibonacci fallible adding machine

If we have a path $(p_0, (a_0/b_0, t_0), p_1) \ldots (p_n, (a_n/b_n, t_n), p_{n+1})$ where $p_0 = I$ and $p_{n+1} = T$, then we say that the number $a_n \ldots a_0$ transitions to the number $b_n \ldots b_0$ and we can remark that the probability of transition is $t_0 t_1 \ldots t_n$.

Example: 10101 transitions to 100000 with probability p^3 and 10101 transitions to 10000 with probability $p(1-p)$.

The transition graph:

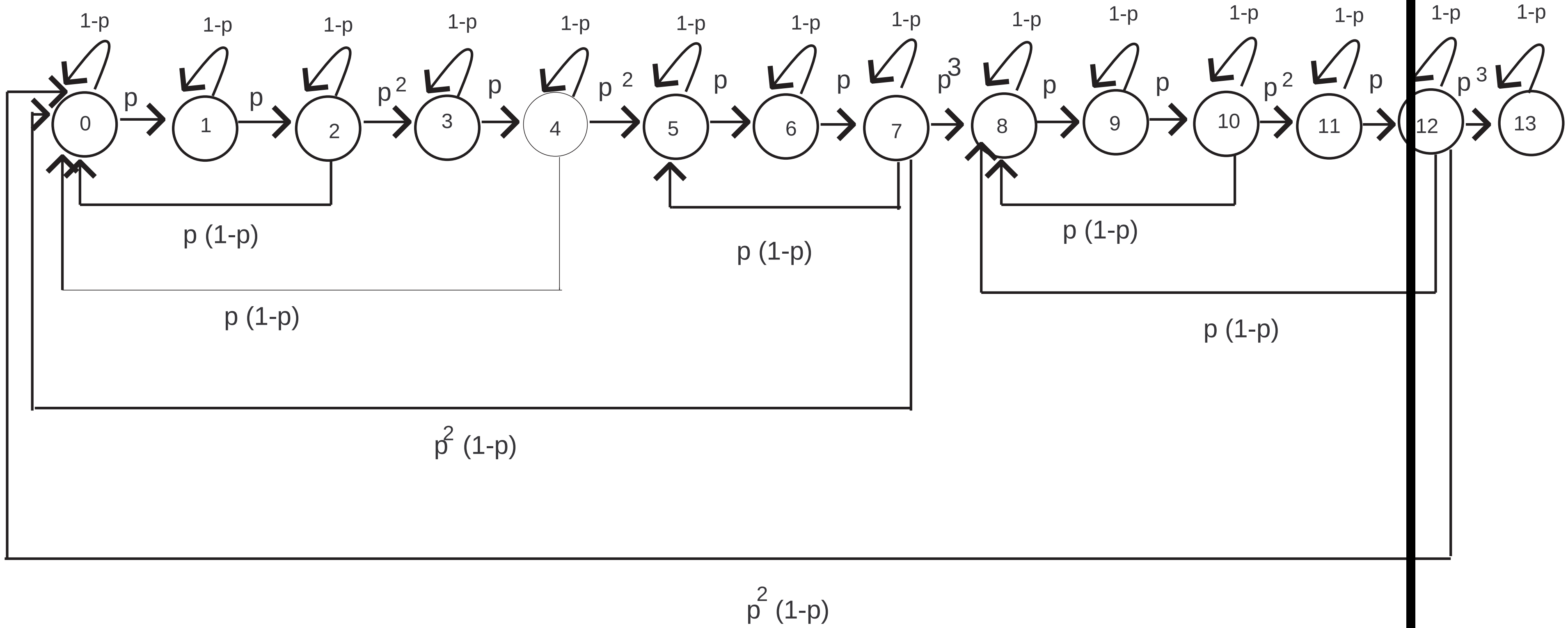

Figure 4: Transition graph of Fibonacci fallible adding machine

The transition operator P is:

$$
P = \begin{pmatrix}
1-p & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 \\
p(1-p) & 0 & 1-p & p^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1-p & p & 0 & 0 & 0 & 0 \\
p(1-p) & 0 & 0 & 0 & 1-p & p^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1-p & p & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1-p & p & 0 \\
p^2(1-p) & 0 & 0 & 0 & 0 & p(1-p) & 0 & 1-p & p^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-p \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p(1-p) & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
$$

Remark 1.1 *The diagonal region of the transition matrix is formed by two blocs A and B where* $A = \begin{pmatrix} 1-p & p & 0 \\ 0 & 1-p & p \\ p(1-p) & 0 & 1-p \end{pmatrix}$ *and* $B = \begin{pmatrix} 1-p & p \\ 0 & 1-p \end{pmatrix}$. *We can prove that the sequence of occurences of A and B is given by the fixed point of Fibonacci substitution.*

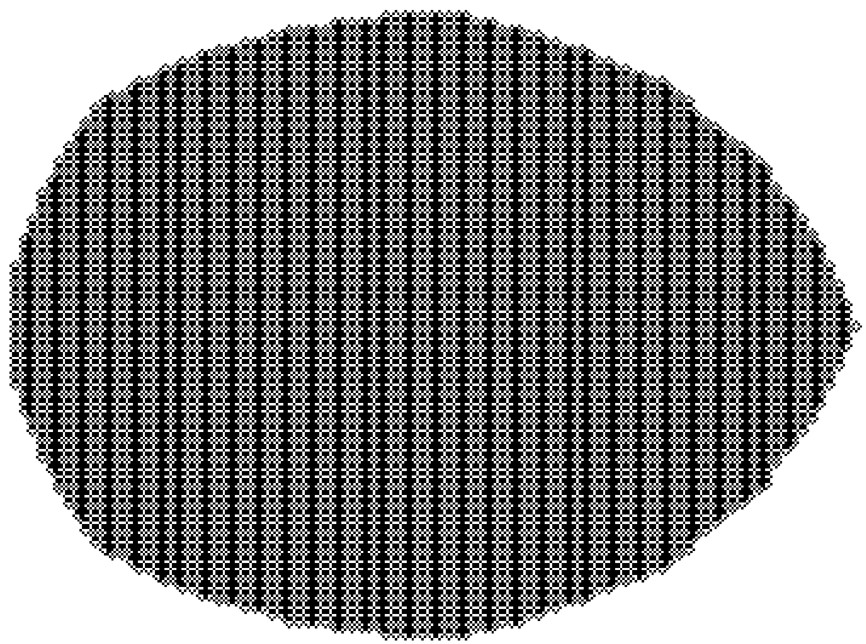

Figure 5: $1/p = 1.1$

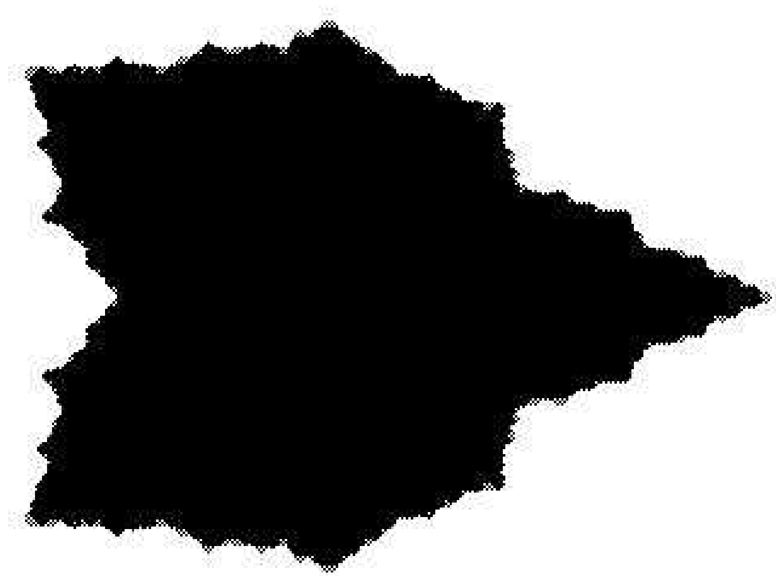

Figure 6: $1/p = 1.5$

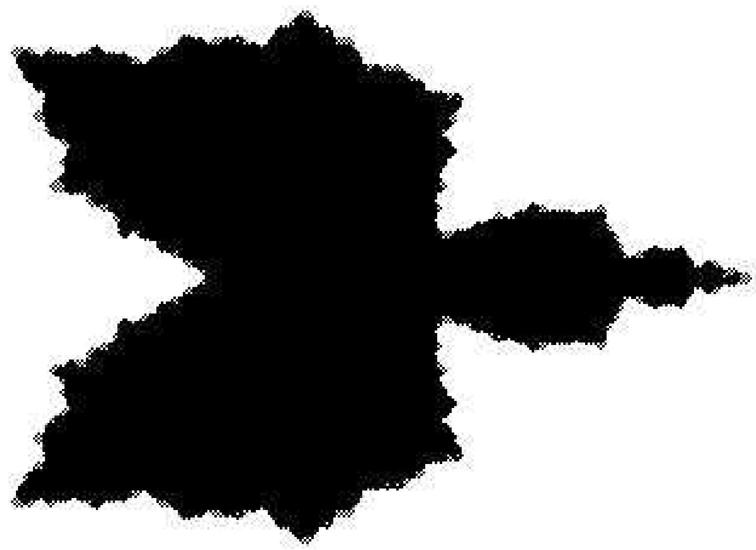

Figure 7: $1/p = 1.6$

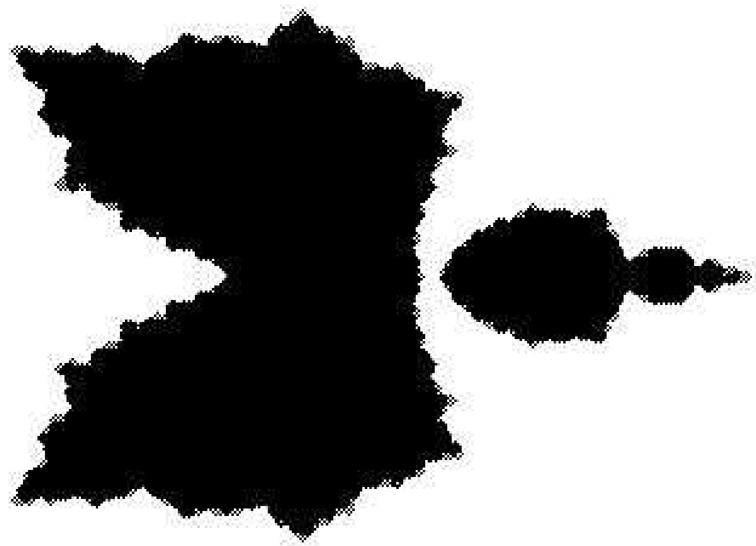

Figure 8: $1/p = 1.61$

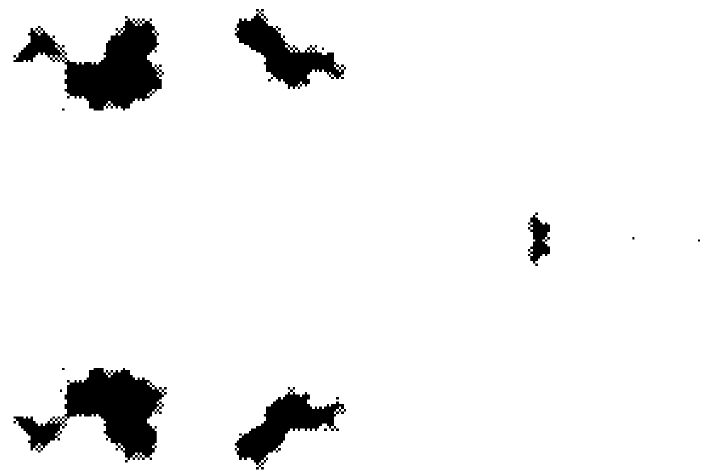

Figure 9: $1/p = 1.8$

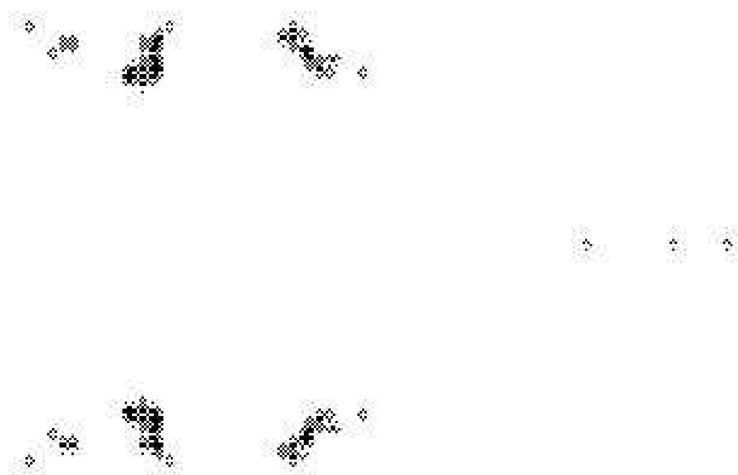

Figure 10: $1/p = 1.9$

Theorem 1 *Let* $f : \mathbb{C}^2 \mapsto \mathbb{C}^2$ *be the function defined by:* $f(x, y) = (\frac{1}{p^2}(x - 1 + p)(y - 1 + p), x)$. *Then the point spectrum of* P, $S(P)$ *in* $l^\infty(\mathbb{N})$ *is contained in the set* $\mathcal{E}_p = \{\lambda \in \mathbb{C} \mid (\lambda_1, \lambda) \in J_f\}$

where J_f *is the filled Julia set of* f *and* $\lambda_1 = 1 - p + \frac{(1-\lambda-p)^2}{p}$.
Moreover

The set $\mathcal{E}_p$ *satisfies the following topological properties.*

1. $\mathbb{C} \setminus \mathcal{E}_p$ *is a connected set.*
2. *If* $0 < p < 1/\beta$ *where* $\beta = \frac{1+\sqrt{5}}{2}$, *then* $\mathcal{E}_p$ *is a disconnected set .*
3. *When* p *converges to* 1 *then* $\mathcal{E}_p$ *is a connected set .*

Conjecture $S(P) = \mathcal{E}_p$

Idea of Proof.

Let λ *be an eigenvalue of* P *associated to the eigenvector* $v = (v_i)_{i \geq 0}$. *Since* $P_{i,i+k} = 0$ *for all* $k \geq 2$ *and* i, *we can prove by induction on* k *that*

$$v_k = q_k(a, \lambda) v_0, \ \forall k \in \mathbb{N} \tag{1}$$

where $a = 1/p$ *and* $q_k(a, \lambda) \in \mathbb{C}$. *We can also prove that for all integer* $n \geq 2$,

$$q_{F_n} = a q_{F_{n-1}} q_{F_{n-2}} - (a-1), \ \forall n \geq 2.$$

where

$$q_{F_0} = q_1 = -\frac{1-\lambda-p}{p}$$

and

$$q_{F_1} = q_2 = \frac{(1-\lambda-p)^2}{p^2}.$$

Let

$$g(x, y) = (axy - (a-1), x).$$

We have

$$(q_{F_k}, q_{F_{k-1}}) = g(q_{F_{k-1}}, q_{F_{k-2}}) = \cdots = g^{k-1}(q_{F_1}, q_{F_0}). \qquad (2)$$

We have $\lambda \in S(P) \Leftrightarrow v_n$ *bounded* $\Leftrightarrow q_n$ *bounded* $\Rightarrow q_{F_n}$ *bounded* $\Rightarrow$ $(q_{F_n}, q_{F_{n-1}})$ *bounded.*

Since

$$(q_{F_n}, q_{F_{n-1}}) = g^{n-1}(q_{F_1}, q_{F_0}) = h^{-1} f^{n-1} h(q_{F_1}, q_{F_0}) = h^{-1} f^{n-1}(\lambda_1, \lambda)$$

Where h *is the* $\mathbb{C}^2$ *map defined by:*

$$h(x, y) = (ax - (a-1), ay - (a-1)).$$

It follows

$$S(P) \subset \{\lambda \in \mathbb{C},\ f^n(\lambda_1, \lambda) \text{ bounded } \} = J(f).$$

□

To give the exact value of $S(P)$, *we need the following lemma.*

Lemma 1 *for all* $0 < k < F_{n-1}$*, we have* $q_{F_n+k} = q_{F_n} q_k$*. In particular for each* $n \in \mathbb{N}$ *with* $n = F_{n_1} + \cdots + F_{n_k}$ *(Fibonacci representation), we have* $q_n = q_{F_{n_1}} \ldots q_{F_{n_k}}$*.*

□

Remark 1.2 *We have*
$S(P) = \{\lambda \in \mathbb{C},\ q_n(\lambda)$ *is bounded for all* $n \in \mathbb{N}\}$ *and*
$\mathcal{E}_p = \{\lambda \in \mathbb{C},\ q_{F_n}(\lambda)$ *is bounded for all* $n \in \mathbb{N}\}$*.*

By doing many computations, we conjecture that $S(P) = \mathcal{E}_p$*.*

1.1 Properties of $\mathcal{E}_p$

Let $(f_n)_{n\geq 0}$ *be the function sequence defined by:*

$$f_0(z) = z,\ f_1(z) = z^2,\ f_n(z) = af_{n-1}(z)f_{n-2}(z) - (a-1),\ \forall n \geq 2,$$

Let

$$\mathcal{K} = \mathcal{K}_p = \{z \in \mathbb{C},\ f_n(z) \text{ is bounded }\} = \{z \in \mathbb{C},\ g^n(z^2, z))\ \text{bounded }\}.$$

It is easy to see that

$$\mathcal{E}_p = \psi(\mathcal{K})$$

where

$$\psi(z) = pz + 1 - p,$$

for all $z \in \mathbb{C}$.

Then we will study properties of $\mathcal{K}$

Proposition 1 *There exists a real number* $R > 1$ *such that if that* $|f_k(z)| > R$ *for some integer* k*, then the sequence* $(f_n(z))_{n \geq 0}$ *is unbounded.*

With this, we obtain that

$$\mathcal{K} = \bigcap_{n=0}^{+\infty} f_n^{-1}\overline{D(0,R)} \textit{ where } \overline{D(0,R)}$$

Working more, we can prove that for all n

$$f_{n+1}^{-1}\overline{D(0,R)} \subset f_n^{-1}\overline{D(0,R)}$$

□

$\mathbb{C} \setminus \mathcal{K}$ **is a connected set**.

Since

$$\mathcal{K} = \bigcap_{n=0}^{+\infty} f_n^{-1}\overline{D(0,R)}$$

, *then*

$$\mathbb{C} \setminus \mathcal{K} = \bigcup_{n=0}^{+\infty} \mathbb{C} \setminus f_n^{-1}\overline{D(0,R)}.$$

We have for all n, $f_n^{-1}\overline{D(0,R)}$ *is a connected set. Since for every* n, $\mathbb{C} \setminus f_n^{-1}\overline{D(0,R)} \subset \mathbb{C} \setminus f_{n+1}^{-1}\overline{D(0,R)}$, *we deduce that* $\mathbb{C} \setminus \mathcal{K}$ *is a connected set.*

$\square$

$\mathcal{K}$ is disconnected if $p > 1/\beta$

Remark 1.3 *This work extends to all Parry sequences* $(F_n)_{n \geq 0}$ *given by the relation*

$$F_{n+d} = a_1 F_{n+d-1} + \cdots + a_d F_n \ \forall n \geq 0,$$

and with initial conditions (Parry conditions)

$$F_0 = 1, \ F_n = a_1 F_{n-1} + \cdots + a_n F_0 + 1 \ \forall 0 \leq n < d,$$

where $a_i, 1 \leq i \leq d$ *are non-negative integers which satisfy the* *relations*

$$a_j a_{j+1} \ldots a_d \leq_{lex} a_1 a_2 \ldots a_{d-j+1} \textit{ for } 2 \leq j \leq d.$$

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