Critical constants for unique expansions in general alphabets

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Journées Numération, Prague 26 - 30 May 2008
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Expansions in alphabets with deleted digits

We consider generic alphabets

$$A = \{a_1, \ldots, a_J\}$$

of real numbers $a_1 < \cdots < a_J$.

Given a real number $q > 1$, by an expansion of a real number x we mean a sequence (c_i) of numbers $c_i \in A$ satisfying the equality

$$\pi_q(c) := \sum_{i=1}^{\infty} \frac{c_i}{q^i} = x.$$

In order to have an expansion, x must belong to the interval $[\frac{a_1}{q-1}, \frac{a_J}{q-1}]$

Interval Condition

This result was proved in [Pedicini2005]:

Theorem

Every $x \in [\frac{a_1}{q-1}, \frac{a_J}{q-1}]$ *has at least one expansion in base* q *if and only if*

$$1 < q \leq Q_A := 1 + \frac{a_J - a_1}{\max_{j>1}\{a_j - a_{j-1}\}} (\leq J). \quad (1)$$

Univoque sequences

Definition

A sequence $(c_i) \in A^\infty$ is called *univoque* in base q if

$$x := \sum_{i=1}^{\infty} \frac{c_i}{q^i}$$

has no other expansion in this base.

Example

The constant sequences $(a_1)^\infty$ and $(a_J)^\infty$ are univoque in every base q: they are called the *trivial unique expansions*.

Characterization theorem

Theorem

Assume the interval condition (1)*. An expansion* (c_i) *is unique in base* q *if and only if the following conditions are satisfied:*

$$\sum_{i=1}^{\infty} \frac{c_{n+i} - a_1}{q^i} < a_{j+1} - a_j \quad \textit{whenever} \quad c_n = a_j < a_J;$$

$$\sum_{i=1}^{\infty} \frac{a_J - c_{n+i}}{q^i} < a_j - a_{j-1} \quad \textit{whenever} \quad c_n = a_j > a_1.$$

Consequences of the characterization

Corollary

For every given set $C \subset A^\infty$ *there exists a number*

$$1 \le q_C \le Q_A$$

such that

$$q > q_C \implies \text{ every sequence } c \in C \text{ is univoque in base } q;$$
$$1 < q < q_C \implies \text{ not every sequence } c \in C \text{ is univoque in base } q.$$

Critical constants

Definition

The number q_C is called the *critical base* of C. If $C = \{c\}$ is a one-point set, then $q_c := q_C$ is also called the critical base of the sequence c.

If C is a nonempty finite set of eventually periodic sequences, then the supremum $\sup q_\alpha$ in the above proof is actually a maximum.

In the last case, it is possible that not all sequences $c \in C$ are univoque in base $q = q_C$.

Generalized Golden number

It is well-known that for the alphabet $A = \{0, 1\}$ there exist nontrivial univoque sequences in base q if and only if $q > \frac{1+\sqrt{5}}{2}$.

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Corollary

There exists a number $1 < G_A \leq Q_A$ such that

$$q > G_A \implies \text{there exist nontrivial univoque sequences;}$$
$$1 < q < G_A \implies \text{there are no nontrivial univoque sequences.}$$

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$$1 < q < G_A \implies \text{there are no nontrivial univoque sequences.}$$

Definition

The number G_A is called the *critical base* of the alphabet A.

Invariance properties

Proposition

The critical base does not change if we replace the alphabet A

- *by* $A + b = \{a_j + b \mid j = 1, \dots, m\}$ *for some real number* b*;*
- *by* $bA = \{ba_j \mid j = 1, \dots, m\}$ *for some nonzero real number* b*;*
- *by the dual alphabet defined by* $D(A) = \{a_m + a_1 - a_j \mid j = 1, \dots, m\}$.

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We wish to establish the critical bases for ternary alphabets

$$A = \{a_1, a_2, a_3\}.$$

Restriction to ternary alphabets

By the Proposition above we may restrict ourselves without loss of generality to the case of alphabets

$$A_m = \{0, 1, m\}$$

with $m \geq 2$.
Interval condition (1) in the ternary case takes the form

$$1 < q \leq \frac{2m-1}{m-1}.$$

Lemma

An expansion (c_i) *is unique in base* q *for the alphabet* A_m *if and only if the following conditions are satisfied:*

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} < 1 \quad \text{whenever } c_n = 0; \tag{2}$$

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} < m-1 \quad \text{whenever } c_n = 1; \tag{3}$$

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} > \frac{m}{q-1} - 1 \quad \text{whenever } c_n = 1; \tag{4}$$

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} > \frac{m}{q-1} - (m-1) \quad \text{whenever } c_n = m. \tag{5}$$

Alphabet $A_3 = \{0, 1, 3\}$

Consider the periodic sequence $(c_i) = (31)^\infty$. By the periodicity of (c_i) we have for each n either $c_n = 3$ and $(c_{n+i}) = (13)^\infty$ or $c_n = 1$ and $(c_{n+i}) = (31)^\infty$.
In this case Theorem 4 contains only three conditions on q.
For $c_n = 3$ we have the condition

$$\sum_{i=1}^{\infty} \frac{3 - c_{n+i}}{q^i} < 2 \Longleftrightarrow \frac{2q}{q^2 - 1} < 2,$$

while for $c_n = 1$ we have the following two conditions:

$$\sum_{i=1}^{\infty} \frac{3 - c_{n+i}}{q^i} < 1 \iff \frac{2}{q^2 - 1} < 1$$

and

$$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q^i} < 2 \iff \frac{3}{q - 1} - \frac{2}{q^2 - 1} < 2.$$

They are equivalent approximatively to the inequalities $q > 1.61803$, $q > 1.73205$ and $q > 2.18614$ respectively, so that $q_c \approx 2.18614$.

Main result

In order to formulate our main result we introduce the quantities $P_m = P(m)$, $p_1 = p_1(k, h, m)$, $p_2 = p_2(k, h, m)$, $p = p(k, h, m)$ for every real number $m > 0$ and nonnegative integers k and h by the following formulae:

$$P(m) = 1 + \sqrt{\frac{m}{m-1}};$$

$$\pi_{p_1}\left(\left(m^k 1 (m^{k-1} 1)^h\right)^\infty\right) = m - 1;$$

$$\pi_{p_2}\left(\left((m^{k-1} 1)^h m^k 1\right)^\infty\right) = \frac{m}{p_2 - 1} - 1;$$

$$p = \max\{p_1, p_2\}.$$

Theorem

If $p(k, h, m) \le P(m)$ for some k and h, then $G_{A_m} = p(k, h, m)$.

Blocks

The blocks $S_{j,m} := m^{j-1}1$ $(j = 1, 2, \ldots)$ have the following property:

Lemma

If $2^k \le m \le 2^{k+1}$ and $1 < q \le P_m$ for some $m \ge 2$, then any nontrivial univoque sequence in base q has either the form

$$0^t 1 S_{\varepsilon_1,m} S_{\varepsilon_2,m} \ldots$$

or the form

$$m^t 1 S_{\varepsilon_1,m} S_{\varepsilon_2,m} \ldots$$

with some nonnegative integer t and some sequence (ε_i) of elements $\varepsilon_i \in \{k, k+1\}$.
Moreover, if $m \ge M \simeq 2.80194$, the largest root of the polynomial $m^3 - 4m^2 + 3m + 1$, then it cannot begin with a 0 digit.

Sketch of proof

Let (c_i) be a univoque sequence in some base $1 < q \leq P_m$. The lemma will follow from the following six properties:

- (c_i) does not contain any block of the form $m0$;
- (c_i) does not contain any block of the form 10;
- (c_i) does not contain any block of the form $0m$;
- if $m > M$, then (c_i) does not contain any block of the form 01;
- each 1 digit is followed by at least $k - 1$ consecutive m digits;
- each 1 digit is followed by at most k consecutive m digits.

Quasi-greedy and quasi-lazy expansions

The quasi-greedy expansion of a real number x in some base q is its lexicographically largest infinite expansion, while the quasi-lazy expansion of x is the conjugate of the quasi-greedy expansion of $\frac{m}{q-1} - x$ with respect to the conjugate alphabet $\{0, m-1, m\}$.

The following characterization of special quasi-greedy and quasi-lazy expansions follows from more general results in [Pedicini2005].

Lemma

Consider a sequence (c_i) with digits in $\{1, m\} \subsetneq A_m$.

Let $q > 1$ and set $x := \sum_{i=1}^{\infty} \frac{c_i}{q^i}$.

If $(c_{i+n}) \leq (c_i)$ whenever $c_n = 1$, then (c_i) is the quasi-greedy expansion of x in base q.

If $(c_{i+n}) \geq (c_i)$ whenever $c_n = 1$, then (c_i) is the quasi-lazy expansion of x in base q.

Our following result explains the introduction of $p = p(k, h, m)$ in Theorem 11.

Lemma

The critical base of the sequence $(S^h_{k,m} S_{k+1,m})^\infty$ *is equal to* $p(k, h, m)$.

Lemma

Assume that $p(k, h, m) \le P_m$. *If* $p_1 \ge p_2$, *then*

$$S_k(S^h_k S_{k+1})^\infty \le (\gamma_i) \quad \textit{and} \quad (\delta_i) = S_{k+1}(S^h_k S_{k+1})^\infty; \tag{6}$$

If $p_2 \ge p_1$, *then*

$$(S^h_k S_{k+1})^\infty = (\gamma_i) \quad \textit{and} \quad (\delta_i) \le m(S^h_k S_{k+1})^\infty. \tag{7}$$

In view of the preceding lemma this implies that

$$S_k(S_k^h S_{k+1})^\infty < (c_{n+i}) < S_{k+1}(S_k^h S_{k+1})^\infty \quad \text{whenever} \quad c_n = 1 \tag{8}$$

if $\pi_p((S_{k+1} S_k^h)^\infty) = m - 1$, and

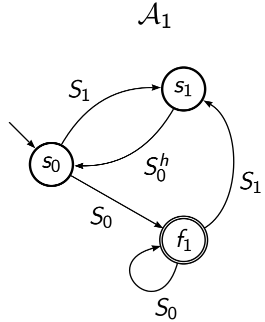
$$(S_k^h S_{k+1})^\infty < (c_{n+i}) < m(S_k^h S_{k+1})^\infty \quad \text{whenever} \quad c_n = 1 \tag{9}$$

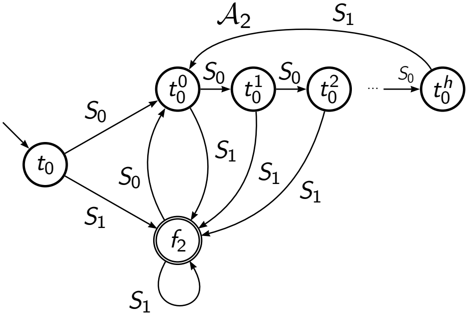
if $\pi_p((S_k^h S_{k+1})^\infty) = \frac{m}{p-1} - 1$.

For any fixed h we can define the following Büchi automata:

- Concerning the right gap $m - 1$:
  - $\mathcal{A}_1$ accepts (c_i) if it satisfies the left-hand side of condition (8);
  - $\mathcal{A}_2$ accepts (c_i) if it satisfies the right-hand side of condition (8);
- Concerning the left gap 1:
  - $\mathcal{A}'_1$ accepts (c_i) if it satisfies the left-hand side of condition (9);
  - $\mathcal{A}'_2$ accepts (c_i) if it satisfies the right-hand side of condition (9).

In these pictures we replaced S_{k+i} by S_i, so that S_0 denotes S_k and S_1 denotes S_{k+1}.





Automata and lexicographic condition (right gap)

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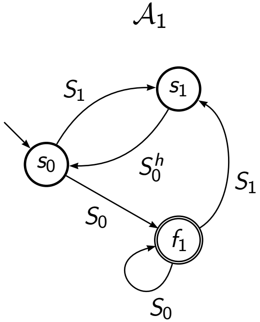
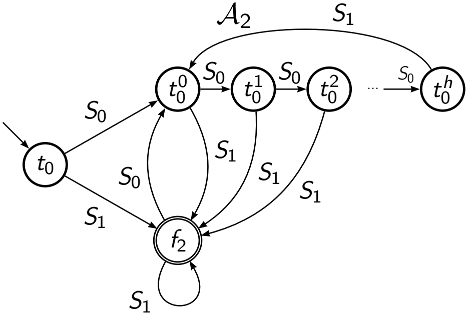
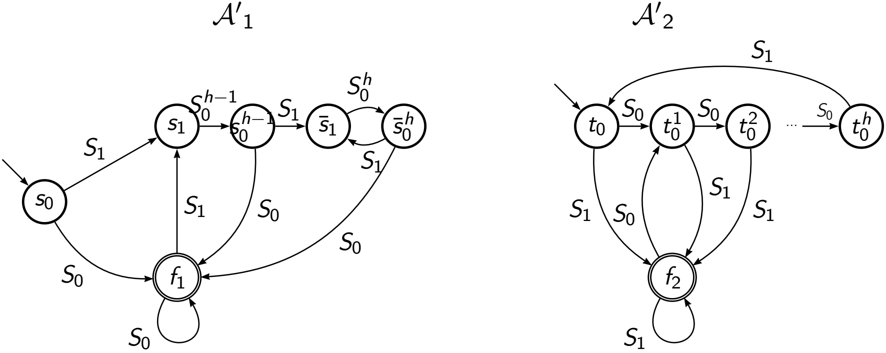


$S_0(S_0^h S_1)^\infty < (c_{n+i})$

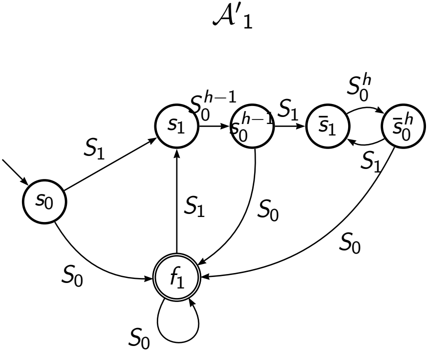


$(c_{n+i}) < S_1(S_0^h S_1)^\infty$

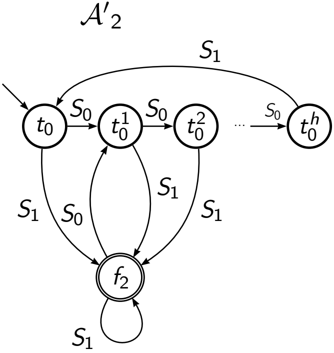
Automata and lexicographic condition (left gap)



Automata and lexicographic condition (left gap)



$(S_0^h S_1)^\infty < (c_{n+i})$



$(c_{n+i}) < m(S_0^h S_1)^\infty$

The proof of the theorem

Note that by Lemma 12 the choice of the alphabet $\{S_k, S_{k+1}\}$ for these automata does not imply a loss of generality.
We can reformulate the conditions (8) and (9) as follows. An expansion (c_i) is unique in base p only if, starting from the smallest n such that in case $\pi_p((S_{k+1}S_k^h)^\infty) = m - 1$

$$(c_{n+i}) \in L(\mathcal{A}_1) \cap L(\mathcal{A}_2) \quad \text{whenever} \quad c_n = 1,$$

while in case $\pi_p((S_k^h S_{k+1})^\infty) = \frac{m}{p-1} - 1$

$$(c_{n+i}) \in L(\mathcal{A}'_1) \cap L(\mathcal{A}'_2) \quad \text{whenever} \quad c_n = 1.$$

Intersection Automaton

The construction of the intersection automaton in the case of infinite words can be obtained A standard construction for the intersection automata in the case of infinite words can be applied:

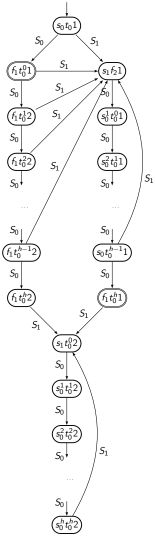
$$\mathcal{A} = (\bar{A}, S \times T \times \{1,2\}, (s_0, t_0, 1), \tau, f_1 \times T \times \{1\}).$$

Denoting by a a generic element of $\bar{A}$, the transition function τ is defined for every $s \in S$ and for every $t \in T$ by the following rules:

$$\begin{aligned}
&\tau((s,t,1),a) = (\tau_1(s,a), \tau(t,a), 1) \quad \text{if } s \neq f_1;\\
&\tau((s,t,2),a) = (\tau_1(s,a), \tau(t,a), 2) \quad \text{if } t \neq f_2;\\
&\tau((f_1,t,1),a) = (\tau_1(f_1,a), \tau(t,a), 2);\\
&\tau((s,f_2,2),a) = (\tau_1(s,a), \tau(f_2,a), 2).
\end{aligned} \tag{10}$$

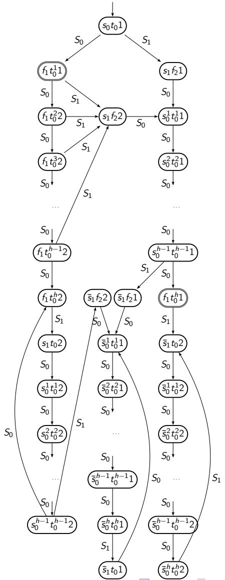
Intersection Automata (right gap)

We build the intersection automaton and we prove that no sequence is accepted, so that there are no other unique sequences in $L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$.



Intersection Automaton (left gap)

The same construction can be done in order to build the intersection automaton for $\mathcal{A}'_1$ and $\mathcal{A}'_2$. Also in this case no sequence is accepted, so that there are no other unique sequences in $L(\mathcal{A}'_1) \cap L(\mathcal{A}'_2)$.



Last part of the work

What we have here is the fact that whenever $p(h,k,m) \leq P(m)$ then for any $q < p(h,k,m)$ we can establish the critical base for such an m.
But when and how it is possible to determine h for a given m ?
The existence proof for such h, k and m is proved in the second part of the work:

Theorem

The condition $p(k,h,m) \leq P(m)$ of Theorem 11 is satisfied for some $k, h \geq 0$ and $m \geq 2$ if and only if $m \in [m_{k,h}, M_{k,h}]$, where $m_{k,h}$ and $M_{k,h}$ are the unique solutions of the equations

$$\pi_{P(m)}\left((S_{k+1,m}S_{k,m}^h)^\infty\right) = m - 1$$

$$\pi_{P(m)}\left((S_{k,m}^h S_{k+1,m})^\infty\right) = \frac{m}{P(m)-1} - 1,$$

respectively.

Golden number

Let $A = \{0, 1\}$ and consider the sequence $(c_i) = 1(10)^\infty$. This sequence is nontrivial and eventually periodic with a nontrivial period, thus there exists a critical base q_c such that $q > q_c$ if and only if (c_i) is unique in base q. By applying the algorithm of the proof of Corollary 5 we obtain that the sequence $\{q_n\}$ is composed of the following elements:

- if $n = 1$ and $c_n = 1$, then $c_{n+i} = (10)^\infty$, so that the equation

  $$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q_1^i} = \frac{1}{q-1} - 1$$

  is equivalent to $q^2 - 2 = 0$ whence $q_1 = \sqrt{2}$.
- if $n = 2$ and $c_n = 1$, then $c_{n+i} = (10)^\infty$, so that the same equation is equivalent to $q^2 - q - 1 = 0$ whence $q_2 = \frac{1+\sqrt{5}}{2}$.
- if $n = 3$ and $c_n = 0$, then $c_{n+i} = (10)^\infty$, and the equation

  $$\sum_{i=1}^{\infty} \frac{c_{n+i}}{q_1^i} = 1$$

A_3

Fix $m = 3$ and consider the sequence $c := (S_1)^\infty = (31)^\infty$.
We recall from Example $A_3 = \{0, 1, 3\}$ that the critical base of this sequence is equal to $q_c \approx 2.18614$.
Since $q_c \leq P_m \approx 2.224744$, applying Theorem 11 we conclude that q_c is the critical base of the alphabet $\{0, 1, 3\}$.

Numerical computation of the intervals of Theorem 16

In the following table we indicate the first five intervals $I_{k,h} := [m_{k,h}, M_{k,h}]$ of Theorem 16,

k	h	$(S_{k+1,m}S^h_{k,m})$	$I_{k,h}$
0	0	$(1)^\infty$	$[1.61803, 2.32471]$
1	3	$(m1111)^\infty$	$[2.34687, 2.37782]$
1	2	$(m111)^\infty$	$[2.37897, 2.46001]$
1	1	$(m11)^\infty$	$[2.46788, 2.72274]$
1	0	$(m1)^\infty$	$[2.80194, 3.56811]$