On the boundary of the closure of the set of contractive polynomials

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Prague, May 28, 2008.

1. Notations and problem setting

Let $\mathbf{r} \in \mathbb{R}^d$. Akiyama, Borbély, Brunotte, Thuswaldner and A.P. introduced in 2005 the mapping
$\tau_{\mathbf{r}} : \mathbb{Z}^d \mapsto \mathbb{Z}^d$ such that if $\mathbf{a} = (a_1, \ldots, a_d)^T \in \mathbb{Z}^d$ then

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \ldots, a_d, -\lfloor \mathbf{r}^T \mathbf{a} \rfloor)^T. \tag{1}$$

For $k \geq 0$ let

$$\tau^k(\mathbf{a}) = \begin{cases} \mathbf{a}, & \text{if} \quad k = 0, \\ \tau(\tau^{k-1}(\mathbf{a})), & \text{if} \quad k > 0 \end{cases}$$

and $a_{d+k+1} = -\lfloor \mathbf{r}^T \tau_{\mathbf{r}}^k(\mathbf{a}) \rfloor$.

$\tau_{\mathbf{r}}$ is closely related to

- canonical number systems
- β-expansions with property (F).

Let

$$\mathcal{D}_d = \{\mathbf{r} \ : \ \{\tau_{\mathbf{r}}^k(\mathbf{a})\}_{k=0}^{\infty} \quad \text{is bounded for all} \quad \mathbf{a} \in \mathbb{Z}^d\}$$

and

$\mathcal{E}_d$: the set of real monic polynomials, whose roots are lying in the closed unit circle.

We proved:

$\mathbf{r} \in \mathcal{D}_d \Longrightarrow R(X) = X^d + r_d X^{d-1} + \ldots + r_2 X + r_1 \in \mathcal{E}_d$

and

if $R(X)$ is lying in the interior of $\mathcal{E}_d$ then $\mathbf{r} \in \mathcal{D}_d$.

Question: What happens if $R(X)$ belongs to the boundary of $\mathcal{E}_d$, i.e. some of its roots are lying on the unit circle? With Turing machine is decidable that $\mathbf{r} \in \mathbb{Q}^d$ belongs to $\mathcal{D}_d$?

The case $d = 2$ was studied by Akiyama, Brunotte, Thuswaldner and A.P. in 2006. $\mathcal{D}_2$ is to the closed triangle with vertices $(-1, 0), (1, -2), (1, 2)$, but
without the points $(1, -2), (1, 2)$,
the line segment $\{(x, -x - 1) : 0 < x < 1\}$ and,
possible, some point of the line segment $\{(1, y) : -2 < y < 2\}$.

Write in the last case $y = 2\cos\alpha$ and $\omega = \cos\alpha + i\sin\alpha$. It is easy to see, that if $y = 0, \pm 1$ (i.e. $\alpha = 0, \pm\pi/2$) then $(1, y) \in \mathcal{D}_2$.
We conjectured: this is true for all points of the line segment.

Akiyama, Brunotte, Steiner and A.P. (2006): the conjecture is true for the golden mean, i.e. for $y = \frac{1+\sqrt{5}}{2}$ and
(2008) for those ω, which are quadratic algebraic numbers.

The case $d \geq 3$. Our most general result is

Theorem 1 *Assume that some t-th roots of unity $\beta_1, \ldots, \beta_s$ are simple zeroes of $R(X)$ and the other zeroes of it have modulus less than one. Then there exist constants c_1 depending on $\beta_1, \ldots, \beta_s$ and c_2 depending on $\beta_1, \ldots, \beta_s$ and $a_1, \ldots, a_d$ such that if $k > c_2$ then*

$$|a_{k+t} - a_k| < c_1.$$

The importance of Theorem 1: c_1 does not depend on $\mathbf{a}$, i.e. the sequence $\{\tau_{\mathbf{r}}^k(\mathbf{a})\}$ is the union of a finite set and finitely many sequences with bounded growth.

Let $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{Z}^d$ and $\bar{\mathbf{1}} = (1, -1, \ldots, (-1)^{d-1})^T \in \mathbb{Z}^d$. As a consequence of Theorem 1 we prove

Theorem 2 *Assume that* 1 *or* -1 *is a simple zero of* $R(X)$ *and the other zeroes of it have modulus less than one. Then there exist a finite set* $A \subset \mathbb{Z}^d$ *with the following property:*
for all $\mathbf{a} \in \mathbb{Z}^d$ *there exist a constant* k *depending on the zeroes of* $R(X)$ *and* $\mathbf{a}$ *and an integer* L *such that* $\{\tau_{\mathbf{r}}^k(\mathbf{a} - L\mathbf{1})\} \in A$ *or* $\{\tau_{\mathbf{r}}^k(\mathbf{a} - L\bar{\mathbf{1}})\} \in A$ *respectively.*

Theorem 2 implies an algorithm to test $\mathbf{r} \in \mathcal{D}_d$ provided 1 or -1 is a simple root of $R(X)$.
Of course we have to test for all $\mathbf{a} \in A$ whether the sequence $\{\tau_{\mathbf{r}}^n(\mathbf{a})\}$ is ultimately periodic or divergent.

2. Preparatory results

Let $\mathbf{P} = \mathbf{P}(\mathbf{r}) \in \mathbb{Z}^{d\times d}$ be the companion matrix of $R(X)$, i.e.

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -r_1 & -r_2 & \dots & -r_d \end{pmatrix}.$$

Then

Lemma 1 *Let* $\mathbf{a} = (a_1, \dots, a_d)^T \in \mathbb{Z}^d$ *and* $1 \le k \in \mathbb{Z}$. *Then there exist* $-1 < \delta_1, \dots, \delta_k \le 0$ *such that*

$$\tau^k(\mathbf{a}) = \mathbf{P}^k \mathbf{a} + \sum_{j=1}^{k} \mathbf{P}^{k-j} \boldsymbol{\delta}_j$$

holds, where $\boldsymbol{\delta}_j = (0, \dots, 0, \delta_j)^T \in \mathbb{R}^d$.

Lemma 2 *Denote* $\beta_1, \ldots, \beta_h$ *the distinct zeroes of* $R(X) = X^d + r_d X^{d-1} + \ldots + r_1$ *with multiplicity* $e_1, \ldots, e_h \geq 1$ *respectively. There exist* $g_{ji}(X), g_i(X) \in \mathbb{Q}(\beta_1, \ldots, \beta_h)[X], 1 \leq i \leq h$ *of degree at most* $e_i - 1$ *such that*

$$a_{k+1} = \sum_{j=1}^{d} a_j \sum_{\ell=1}^{h} g_{j\ell}(k)\beta_\ell^k + \sum_{j=1}^{k} \delta_j \sum_{\ell=1}^{h} g_\ell(k-j)\beta_\ell^{k-j}. \qquad (2)$$

3. Proof of Theorems 1 and 2

Let $t \geq 1$. Then equation (2) implies

$$\begin{aligned} a_{k+t+1} - a_{k+1} &= \sum_{\ell=1}^{h} \beta_\ell^k \sum_{j=1}^{d} a_j \left(\beta_\ell^t g_{1j\ell}(k+t) - g_{1j\ell}(k) \right) \\ &+ \sum_{j=k+1}^{k+t} \delta_j \sum_{\ell=1}^{h} g_\ell(k+t-j) \beta_\ell^{k+t-j} \\ &+ \sum_{j=1}^{k} \delta_j \sum_{\ell=1}^{h} \beta_\ell^{k-j} \left(g_\ell(k+t-j)\beta_\ell^t - g_\ell(k-j) \right). \end{aligned}$$

The assumptions imply for all $\ell = 1, \ldots, s, j = 1, \ldots, d$

$$\beta_\ell^t g_{1j\ell}(k+t) - g_{1j\ell}(k) = g_\ell(k+t-j)\beta_\ell^t - g_\ell(k-j) = 0.$$

Thus our expression simplifies to

$$\begin{aligned}|a_{k+t+1} - a_{k+1}| \;\leq\; & \left|\sum_{\ell=s+1}^{h} \beta_\ell^k \sum_{j=1}^{d} a_j \left(\beta_\ell^t g_{1j\ell}(k+t) - g_{1j\ell}(k)\right)\right| \\ & + \left|\sum_{j=k+1}^{k+t} \delta_j \sum_{\ell=1}^{h} g_\ell(k+t-j)\beta_\ell^{k+t-j}\right| \\ & + \left|\sum_{j=1}^{k} \delta_j \sum_{\ell=s+1}^{h} \beta_\ell^{k-j} \left(g_\ell(k+t-j)\beta_\ell^t - g_\ell(k-j)\right)\right|.\end{aligned}$$

Changing j to $j+k$ we estimate the second summand as follows

$$\left|\sum_{j=1}^{t} \delta_{j+k} \sum_{\ell=1}^{h} g_\ell(t-j)\beta^{t-j}\right| \le \sum_{j=0}^{t-1} \sum_{\ell=1}^{h} |g_\ell(j)|.$$

As $|\beta_\ell| < 1$ for $\ell = s+1, \ldots, h$ and $|\delta_j| < 1$ for $j = 1, \ldots, k$ there exists a constant $c_2 = c_2(R, \mathbf{a})$ such that if $k \ge c_2$ then

$$\left|\beta_\ell^k \sum_{j=1}^{d} a_j \left(g_{1j\ell}(k+t)\beta_\ell^t - g_{1j\ell}(k)\right)\right| < \frac{1}{2h}$$

By the same reason there exists a constant $c_3(R)$ such that if $k \ge c_3$ then

$$\left|\sum_{\ell=s+1}^{h} \beta_\ell^k \left(g_\ell(k+t)\beta_\ell^t - g_\ell(k)\right)\right| < |\beta_\ell|^{k/2}.$$

Thus

$$
\begin{aligned}
|a_{k+t+1} - a_{k+1}| \;\leq\; & 1/2 + \sum_{j=0}^{t-1} \sum_{\ell=1}^{h} |g_\ell(j)| \\
& + \left| \sum_{j=1}^{k-c_2} \delta_j \sum_{\ell=s+1}^{h} \beta_\ell^{k-j} \Big(g_\ell(k+t-j)\beta_\ell^t - g_\ell(k-j) \Big) \right| \\
& + \left| \sum_{j=k-c_2+1}^{k} \delta_j \sum_{\ell=s+1}^{h} \beta_\ell^{k-j} \Big(g_\ell(k+t-j)\beta_\ell^t - g_\ell(k-j) \Big) \right|.
\end{aligned}
$$

The third summand is bounded by

$$\sum_{j=0}^{\infty} |\beta_\ell^{j/2}| = \frac{1}{1 - |\beta_\ell^{1/2}|},$$

while the fourth summand can be estimated as above and we get for it the upper bound

$$\sum_{j=0}^{c_2-1} \sum_{\ell=s+1}^{h} \left| g_\ell(t+j)\beta_\ell^t - g_\ell(j) \right|,$$

which is a constant depending only on the roots of $R(X)$. The sum of these bounds depends only on the roots of $R(X)$ and we can choose it as c_1. $\square$

Proof of Theorem 2. Let $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{Z}^d$ and $\bar{\mathbf{1}} = (1, -1, \ldots, (-1)^{d-1})^T \in \mathbb{Z}^d$.
If $R(1) = 0$ then $\mathbf{r}^T\mathbf{1} = r_1 + \ldots + r_d = -1$, thus $\tau_{\mathbf{r}}(\mathbf{1}) = \mathbf{1}$. Let n be an integer, then $\mathbf{r}^T(n\mathbf{1}) = nr_1 + \ldots + nr_d = -n$, thus $\tau_{\mathbf{r}}(n\mathbf{1}) = n\mathbf{1}$, i.e $(n\mathbf{1})$ is a fixed point of $\tau_{\mathbf{r}}$ for all integers n.

By Theorem 1 there exist a constant c_1 such that if $k > c_2$ then $|a_{k+1} - a_k| < c_1$. Fix such a k and consider $a_{k+i}, i = 0, \ldots, d-1$ of $\{a_n\}$. Put $L = \min\{a_{k+i}, i = 0, \ldots, d-1\}$ and assume that $L = a_{k+j}$. If $h \in [0, d-1]$ then $0 \leq a_{k+h} - L \leq (d-1)c_1$. To prove the upper bound assume that $h > j$. Then

$$\begin{aligned} a_{k+h} - L &= a_{k+h} - a_{k+j} = a_{k+h} - a_{k+h-1} + \ldots + a_{k+j+1} - a_{k+j} \\ &\leq |a_{k+h} - a_{k+h-1}| + \ldots + |a_{k+j+1} - a_{k+j}| \\ &\leq (d-1)c_1. \end{aligned}$$

The case $h < j$ can be handled similarly.

Let $\mathbf{b} = \mathbf{a} - L\mathbf{1}$. Then we have

$$\tau_{\mathbf{r}}^{u}(\mathbf{b}) = \tau_{\mathbf{r}}^{u}(\mathbf{a}) - \tau_{\mathbf{r}}^{u}(L\mathbf{1}) = \tau_{\mathbf{r}}^{u}(\mathbf{a}) - L\mathbf{1}$$

for all $u \geq 0$. Putting $u = k-1$ we get $\tau_{\mathbf{r}}^{k-1}(\mathbf{a} - L\mathbf{1}) = \tau_{\mathbf{r}}^{k-1}(\mathbf{a}) - L\mathbf{1} = (a_k - L, \ldots, a_{k+d-1} - L)$. Thus the set $A = \{0, \ldots, (d-1)c_1\}^d$ satisfies the assertion.

The proof of the second assertion is similar. $\square$

3. The case $d = 3$

Let $\{G_n\}_{n=0}^{\infty}$ be the linear recurring sequence defined by the initial terms $G_0 = G_1 = 0$, $G_2 = 1$ and by

$$G_{n+3} = -r_3 G_{n+2} - r_2 G_{n+1} - r_1 G_n, \quad n \geq 0. \tag{3}$$

Lemma 2 implies

$$a_{k+1} = -r_1 G_{k-1} a_1 + (G_{k+1} + r_3 G_k) a_2 + G_k a_3 + \sum_{j=1}^{k} G_{k-j} \delta_j. \tag{4}$$

We use the following result of M. Ward:

Lemma 3 *Let the linear recurring sequence* $\{G_n\}_{n=0}^{\infty}$ *be defined by* (3)*. Assume that* $R(x)$ *is square-free and denote* $\alpha_1, \ldots, \alpha_d$ *its roots. Then*

$$G_n = \sum_{h=1}^{d} \frac{\alpha_h^n}{R'(\alpha_h)},$$

where $R'(X)$ *denotes the derivatives of* $R(X)$*.*

By a recent result of Paul Surer the boundary of $\mathcal{E}_3$ is the union of the sets $B_1 = \{(-s, s-(s+1)t, (s+1)t-1) \ : \ -1 \leq s, t \leq 1\}$, $B_2 = \{(s, -s-(s+1)t, (s+1)t+1) \ : \ -1 \leq s, t \leq 1\}$ and $B_3 = \{(v, 1+2tv, 2t+v) \ : \ -1 \leq t, v \leq 1\}$.

3.1 The set B_1

In this case

$$\begin{aligned} R(X) &= X^3 + ((s+1)t-1)X^2 + (s-(s+1)t)X - s \\ &= (X-1)(X^2 + (s+1)tX + s) \\ &= (X-1)(X-\alpha)(X-\beta). \end{aligned}$$

Using Lemma 3 we get

$$\begin{aligned} G_n &= \frac{1}{R'(1)} + \frac{\alpha^n}{R'(\alpha)} + \frac{\beta^n}{R'(\beta)} \\ &= \frac{1}{(s+1)(t+1)} \left(1 + \frac{\alpha^n(\beta-1) - \beta^n(\alpha-1)}{\alpha-\beta}\right). \end{aligned}$$

After some computation we prove

Theorem 3 *Assume that* $\mathbf{r} = (-s, s-(s+1)t, (s+1)t-1), -1 < s, t < 1$*. Let* α, β *be the roots of* $R(X) = X^3 + ((s+1)t-1)X^2 + (s-(s+1)t)X - s$*, which have modulus less than* 1*. Let*

$$c_{11} = \left\lfloor \frac{1}{|\alpha-\beta|}\left(\frac{1}{1-|\alpha|} + \frac{1}{1-|\beta|}\right)\right\rfloor$$

and $A = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \ : \ 0 \leq x_1 \leq c_{11}, x_1 - c_{11} \leq x_2 \leq x_1 + c_{11}, x_2 - c_{11} \leq x_3 \leq x_2 + c_{11}\}$*. There exist for any* $(a_1, a_2, a_3) \in \mathbb{Z}^3$ *integers* L, k *such that* $\tau_{\mathbf{r}}^k(a_1 - L, a_2 - L, a_3 - L) \in A$*.*

Theorem 4 *Assume that* $\mathbf{r} = (-s, s-(s+1)t, (s+1)t-1)$, $-1 < s, t < 1$ *and put* $u = (s+1)t$.

- *If* $u < -s$ *and* $\mathbf{a} = (0,1,2)$ *then* $a_{n+1} = a_n + 1$ *holds for all* $n \geq 0$.
- *If* $u \geq -s$ *and* $s < 0$ *and* $\mathbf{a} = (0,0,1)$ *then* $a_3 = 1$ *and* $a_{n+2} = a_n + 1$ *holds for all* $n \geq 0$.
- *If* $\frac{s+2}{2} < u < \frac{2s+3}{3}$ *and* $s > 3/4$ *and* $\mathbf{a} = (0,1,2)$ *then* $a_3 = 0, a_4 = 3$ *and* $a_{n+5} = a_n + 1$ *holds for all* $n \geq 0$.
- *If* $\frac{4s+5}{5} < u < 3s-1$ *and* $s > 3/4$? *and* $\mathbf{a} = (0,5,0)$ *then* $a_3 = 5, a_4 = 1, a_5 = 3, a_6 = 2, a_7 = 1, a_8 = 4$ *and* $a_{n+9} = a_n + 1$ *holds for all* $n \geq 0$.
- *If* $\frac{3s+4}{4} < u < \min\{\frac{4s+5}{5}, 3s-1\}$ *and* $s > 3/4$? *and* $\mathbf{a} = (0,5,1)$ *then* $a_3 = 3, a_4 = 2, a_5 = 1, a_6 = 4$ *and* $a_{n+7} = a_n + 1$ *holds for all* $n \geq 0$.

In the above cases $\mathbf{r}$ *does not belong to* $\mathcal{D}_3$.

The proof is a simple induction.

In contrast to the last theorem we prove in the next theorem that large portions of $\mathcal{B}_1$ belong to $\mathcal{D}_3$.

Theorem 5 *Assume that* $\mathbf{r} = (-s, s-(s+1)t, (s+1)t-1)$, $-1 < s, t < 1$. *If*

- $-s, s-(s+1)t, (s+1)t-1) \leq 0$ *or*
- $s \in (0.334, 0.468)$ *and* $t = -\frac{s}{s+1}$

then there exist for any $\mathbf{a} \in \mathbb{Z}^3$ *the sequence* $\{\tau_{\mathbf{r}}^k(\mathbf{a})\}$ *is ultimately constant, i.e.* $\mathbf{r} \in \mathcal{D}_3$.

(1) As $\tau_{\mathbf{r}}^{k}(\mathbf{a}+L\mathbf{1}) = \tau_{\mathbf{r}}^{k}(\mathbf{a}) + L\mathbf{1}$ we may assume w.l.g that all coordinates of $\mathbf{a}$ are non negative.

Now assume that $a_{n-1}, a_n, a_{n+1} \geq 0$ for some $n \geq 1$. Then

$$\begin{aligned} -\max\{a_{n-1}, a_n, a_{n+1}\} &\leq -sa_{n-1} + (s-(s+1)t)a_n + ((s+1)t-1)a_{n+1} \\ &\leq -\min\{a_{n-1}, a_n, a_{n+1}\} \end{aligned}$$

and equality holds iff $a_{n-1} = a_n = a_{n+1}$, in which case we are done. Otherwise,

$$\min\{a_{n-1}, a_n, a_{n+1}\} + 1 \leq a_{n+2} \leq \max\{a_{n-1}, a_n, a_{n+1}\},$$

i.e., the minimum of three consecutive terms is increasing, but their maximum is not, thus the sequence becomes constant after some steps.

(2) In this case we apply Theorem 3. We have $R(X) = (X - 1)(X^2 - sX + s)$. The roots α, β of $X^2 - sX + s$ are for $0 \le s \le 1$ conjugate complex numbers, hence $|\alpha| = |\beta| = \sqrt{s}$. Theorem 3 implies $c_{11}(s) = \frac{2}{(1-\sqrt{s})\sqrt{4s-s^2}}$.

Now $c_{11}(s) \ge 4$ for $s \in (0,1)$ and
$c_{11}(s) < 5$ if $s \in (0.079, 0.478)$.
$A(4) = \{(x_1, x_2, x_3) \,:\, 0 \le x_1 \le 4, |x_2 - x_1| \le 4, |x_3 - x_2| \le 4\}$.

For $\mathbf{a} \in A(4)$ we tested the sequence $\{a_n\}$ for $s \in (0.334, 0.468)$.
Convexity property: If $\tau_{\mathbf{r_1}}(\mathbf{a}) = \tau_{\mathbf{r_2}}(\mathbf{a})$ then $\tau_{\mathbf{r}}(\mathbf{a}) = \tau_{\mathbf{r_2}}(\mathbf{a})$ for all $r = \mu r_1 + \nu r_2, 0 \leq \mu, \nu \leq 1$.

This allows to test only the end points of the interval. Using MAPLE 9 we found that $\tau_{\mathbf{r}(0.334)}(\mathbf{a}) = \tau_{\mathbf{r}(0.468)}(\mathbf{a})$ except when $\mathbf{a} = (0, 4, 0), (0, -4, 0)$.
For $\mathbf{a} = (0, -4, 0)$
$\{a_n\} = (0, -4, 0, 3, 3, 2, 2, 3, 4, 4, 4)$, if $0.334 \leq s \leq 0.375$
$\{a_n\} = (0, -4, 0, 4, 4, 3, 3, 4, 5, 5, 5)$, if $0.375 < s \leq 0.468$.
For $\mathbf{a} = (0, 4, 0)$
$\{a_n\} = (0, 4, 0, -2, -1, 1, 2, 2, 2)$, if $0.334 \leq s < 0.375$
$\{a_n\} = (0, 4, 0, -3, -2, 0, 1, 1, 1)$, if $0.375 \leq s \leq 0.468$.
This completes the proof. $\square$

3.2 The set B_2

In this case $R(X) = X^3 + ((s+1)t+1)X^2 - (s+(s+1)t)X + s = (X+1)(X^2 + (s+1)tX + s) = (X+1)(X-\alpha)(X-\beta)$.

Theorem 6 *Assume that* $\mathbf{r} = (s, -(s+1)t - s, (s+1)t + 1)$, $-1 < s, t < 1$ *and put* $u = (s+1)t$.

- *If* $-1 < s \leq 0$ *and* $t > 0$, *but* $(s,t) \neq (-1,1)$ *and* $\mathbf{a} = (0,0,1)$, *then* $a_{2n+f} = (-1)^f n, n = 0, 1, \ldots, f = 0, 1$.
- *If* $s \leq 0$ *and* $1 + 2s < u < 1 + \frac{1}{2}$ *and* $\mathbf{a} = (0, -1, 3)$ *then* $a_3 = -4, a_4 = 6, a_5 = -7$ *and* $a_{n+6} = a_n + 9$ *holds for all* $n \geq 0$.
- *If* $s, u + 1 \geq 0$, *but* $s + u < 0$ *then the sequence* $\{a_n\}$ *is for all initial vectors ultimately periodic with period* $L, -L$ *for some integer* L.

In cases (1) *and* (2) $\mathbf{r}$ *does not belong to* $\mathcal{D}_3$, *while in case* (3) *it does belong.*

The proof is similar, as for B_2.