Sharp bounds for (a)symmetric Diophantine approximation.

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Outline

1. Continued fractions and their approximation qualities
2. The natural extension of continued fractions
3. Sharp bounds

Definition

Every $x \in \mathbb{R} \setminus \mathbb{Q}$ has a unique and infinite regular continued fraction expansion

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}} = [a_0; a_1, a_2, \ldots, a_n, \ldots],$$

where $a_i \in \mathbb{N}$. From now on we assume $a_0 = 0$. The partial coefficients a_i can be found from the regular continued fraction map $T : [0,1) \to [0,1)$

$$T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, x \neq 0; \quad T(0) := 0.$$

Approximations

Finite truncation yields convergents

$$\frac{p_n}{q_n} = [0; a_1, a_2, \ldots, a_n].$$

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}, \quad n \geq 0.$$

Approximation quality

The *approximation coefficients* Θ_n of x are defined by

$$\Theta_n = q_n^2 \left| x - \frac{p_n}{q_n} \right|, \qquad \text{for } n \geq 0. \tag{1}$$

Theorem (Borel, 1903)

For all $n \geq 1$

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < \frac{1}{\sqrt{5}}.$$

Other approximation measures

Dirichlet

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}, \qquad \text{for } n \geq 0,$$

Define the sequences of irrationals C_n and D_n, $n \geq 0$, by

$$\begin{aligned} x - \frac{p_n}{q_n} &= \frac{(-1)^n}{C_n q_n q_{n+1}}, \\ D_n &= [a_{n+1}; a_n, \ldots, a_1] \cdot [a_{n+2}; a_{n+3}, \ldots]. \end{aligned}$$

It is not hard to show that $C_n = 1 + \dfrac{1}{D_n}$.

Question

Suppose $D_{n-2} < r$ and $D_n < R$ for some given reals $r, R > 1$, what can we say about D_{n-1}?

Theorem (Tong, 2004)

If $r > 1$ and $R > 1$ are two real numbers and

$$M_{\mathrm{Tong}} = \frac{1}{2} \left(\frac{1}{r} + \frac{1}{R} + a_n a_{n+1} \left(1 + \frac{1}{r}\right)\left(1 + \frac{1}{R}\right) + \sqrt{\left[\frac{1}{r} + \frac{1}{R} + a_n a_{n+1} \left(1 + \frac{1}{r}\right)\left(1 + \frac{1}{R}\right)\right]^2 - \frac{4}{rR}} \right),$$

then

1. $D_{n-2} < r$ and $D_n < R$ imply $D_{n-1} > M_{\mathrm{Tong}}$;
2. $D_{n-2} > r$ and $D_n > R$ imply $D_{n-1} < M_{\mathrm{Tong}}$.

The future and past

Write T_n and V_n for the "future" respectively "past" of $\frac{p_n}{q_n}$,

$$T_n = [0; a_{n+1}, a_{n+2}, \dots] \quad \text{and} \quad V_n = [0; a_n, \dots, a_1].$$

Define $\Omega = [0,1) \times [0,1]$ and $\mathcal{T} : \Omega \to \Omega$ as

$$\mathcal{T}(x,y) = \left(\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \frac{1}{a_1 + y} \right).$$

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$$\mathcal{T}(x,y) = \left(\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \frac{1}{a_1 + y} \right).$$

We have $(T_n, V_n) = \mathcal{T}^n(x, 0)$.

Rewriting D_{n-2}, D_{n-1} and D_n.

We can expres D_{n-2}, D_{n-1} and D_n in terms of a_n, a_{n+1}, T_n and V_n.

$$D_{n-2} = \frac{(a_n + T_n)V_n}{1 - a_n V_n}, \quad D_{n-1} = \frac{1}{T_n V_n}, \text{ and } D_n = \frac{(a_{n+1} + V_n)T_n}{1 - a_{n+1} T_n},$$

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$$\begin{aligned} D_{n-1} &= [a_n; a_{n-1}, \ldots, a_1] \cdot [a_{n+1}; a_{n+2}, \ldots] \\ &= \left(a_n + \frac{1}{a_{n-1} + \frac{1}{a_{n-2} + \cdots + \frac{1}{a_1}}}\right) \cdot \left(a_{n+1} + \frac{1}{a_{n+2} + \frac{1}{a_{n+3} + \ldots}}\right) \\ &= \frac{1}{V_n}\frac{1}{T_n}. \end{aligned}$$

With

$$D_{n-2} = \frac{(a_n + T_n)V_n}{1 - a_n V_n} \text{ and } D_n = \frac{(a_{n+1} + V_n)T_n}{1 - a_{n+1}T_n},$$

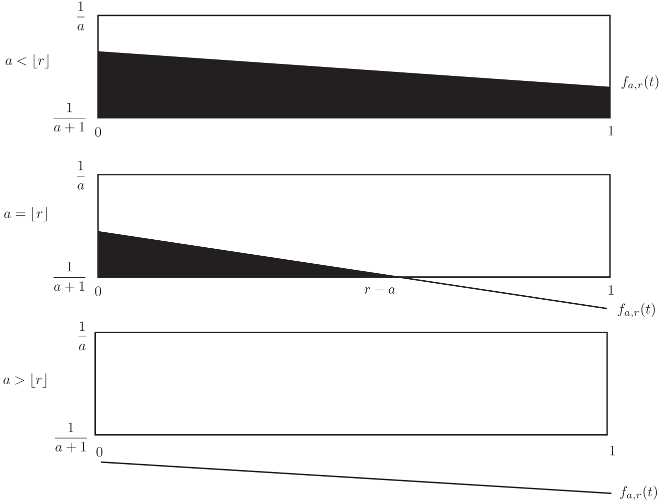
we find

$$\begin{aligned} D_{n-2} < r \qquad & \text{if and only if} \qquad v < f_{a,r}(t), \\ D_n < R \qquad & \text{if and only if} \qquad v < g_{b,R}(t), \end{aligned}$$

with

$$f_{a,r}(t) = \frac{r}{a(r+1)+t} \quad \text{and} \quad g_{b,R}(t) = \frac{R}{t} - b(R+1).$$

The position of the graph of $f_{a,r}(t) = \frac{r}{a(r+1)+t}$ in a horizontal strip with $a_n = a$.



Example with $D_{n-2} < 2.9$ and $D_n < 3.6$.

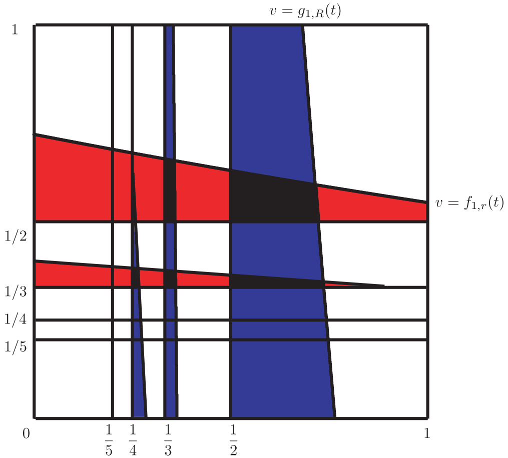


Figure: The regions where $D_{n-2} < 2.9$ are red, the regions where $D_n < 3.6$ are blue. The intersection where both $D_{n-2} < 2.9$ and $D_n < 3.6$ is black.

Three possible configurations of the graphs of f and g when $D_{n-2} < r$ and $D_n < R$ on $\left[\frac{1}{b+1}, \frac{1}{b}\right) \times \left[\frac{1}{a+1}, \frac{1}{a}\right)$.

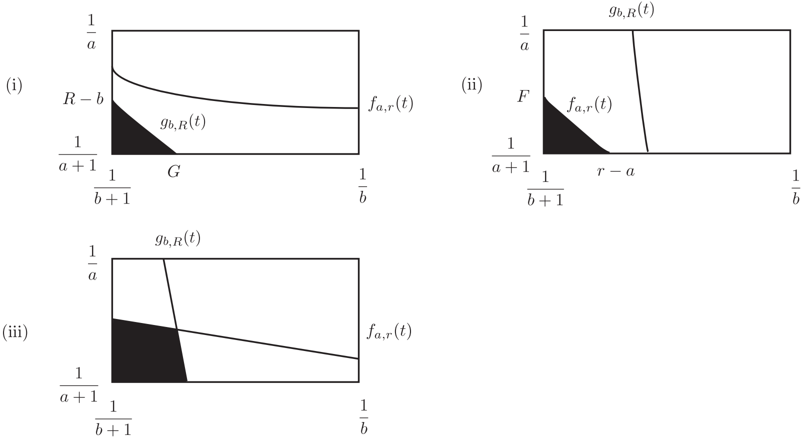


Figure: The part where both $D_{n-2} < r$ and $D_n < R$ is black. Use that $D_{n-1} = \frac{1}{TV}$ find the minimum for D_{n-1}.

Theorem, 2008

Let $r, R > 1$ be reals. If $D_{n-2} < r$ and $D_n < R$, then there are three possibilities for the minimum of D_{n-1}.

1. If $r - a_n \geq G$ and $\frac{1}{a_n + 1} \leq R - a_{n+1} < F$, then
$$D_{n-1} > \min\left\{\frac{a_{n+1}+1}{R - a_{n+1}}, \frac{a_n + 1}{G}\right\}.$$
2. If $\frac{1}{a_{n+1}+1} \leq r - a_n < G \text{ and } R - a_{n+1} \geq F$, then
$$D_{n-1} > \min\left\{\frac{a_{n+1}+1}{F}, \frac{a_n + 1}{r - a_n}\right\}$$
3. In all other cases
$$D_{n-1} > M_{\text{Tong}}.$$

These bounds are sharp.

Example with $D_{n-2} < 2.9$ and $D_n < 3.6$.

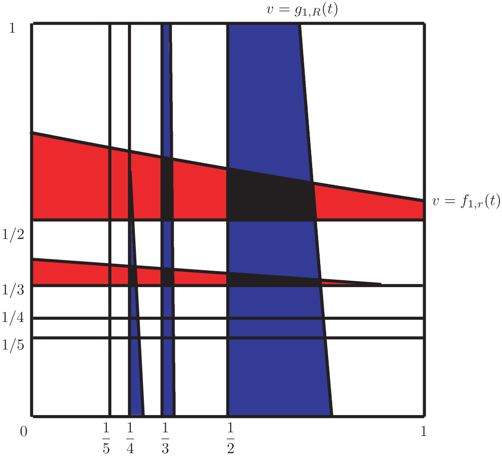


Figure: The regions where $D_{n-2} < 2.9$ are red, the regions where $D_n < 3.6$ are blue. The intersection where both $D_{n-2} < 2.9$ and $D_n < 3.6$ is black.

Example with $D_{n-2} < 2.9$ and $D_n < 3.6$ (continued).

The sharp bounds for this example

$$a_n = 1 \text{ and } a_{n+1} = 1 : D_{n-1} > M_{\mathrm{Tong}} \approx 2.30.$$
$$a_n = 1 \text{ and } a_{n+1} = 2 : D_{n-1} > M_{\mathrm{Tong}} \approx 4.04.$$
$$a_n = 2 \text{ and } a_{n+1} = 1 : D_{n-1} > M_{\mathrm{Tong}} \approx 4.04.$$
$$a_n = 2 \text{ and } a_{n+1} = 2 : D_{n-1} > M_{\mathrm{Tong}} \approx 7.48.$$
$$a_n = 2 \text{ and } a_{n+1} = 3 : D_{n-1} > M_{\mathrm{Tong}} \approx 10.92.$$

Example with $D_{n-2} < 2.9$ and $D_n < 3.6$ (continued).

The sharp bounds for this example

$$\begin{aligned} a_n = 1 \text{ and } a_{n+1} = 1 : D_{n-1} > M_{\text{Tong}} &\approx 2.30. \\ a_n = 1 \text{ and } a_{n+1} = 2 : D_{n-1} > M_{\text{Tong}} &\approx 4.04. \\ a_n = 2 \text{ and } a_{n+1} = 1 : D_{n-1} > M_{\text{Tong}} &\approx 4.04. \\ a_n = 2 \text{ and } a_{n+1} = 2 : D_{n-1} > M_{\text{Tong}} &\approx 7.48. \\ a_n = 2 \text{ and } a_{n+1} = 3 : D_{n-1} > M_{\text{Tong}} &\approx 10.92. \end{aligned}$$

If $a_n = 1$ and $a_{n+1} = 3$, then

$$\begin{aligned} D_{n-1} &> \min\left\{\frac{a_{n+1}+1}{R-a_{n+1}}, \frac{(a_n+1)a_{n+1}(R+1)+1}{R}\right\} \\ &\approx \min\{6.67, 7.94\} = 6.67 \gg M_{\text{Tong}} \approx 5.76. \end{aligned}$$

Example with $D_{n-2} < 2.9$ and $D_n < 3.6$ (continued).

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\end{aligned}$$

For all other values of a_n and a_{n+1} either $D_{n-2} > 2.9$ or $D_n > 3.6$, or both.

Example $D_{n-2} > 2.9$ and $D_n > 3.6$

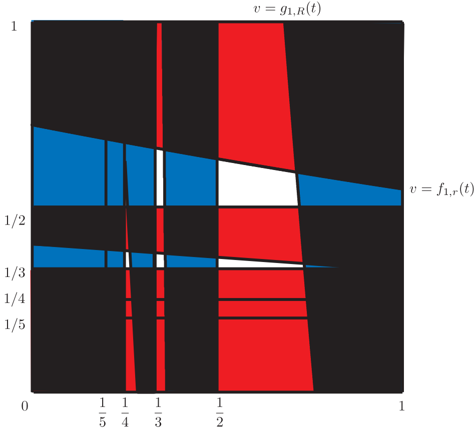

Figure: The regions where $D_{n-2} > r$ are red, the regions where $D_n > R$ are blue. The intersection where both $D_{n-2} > r$ and $D_n > R$ is black.

Example asymptotic frequencies

For almost all $x \in [0,1) \setminus \mathbb{Q}$ the asymptotic frequency that simultaneously $D_{n-2} > 2.9$ and $D_n > 3.6$ is 0.619.

Given that $D_{n-2} > 2.9$ and $D_n > 3.6$ the conditional probability that M_{Tong} is the sharp bound is 0.28.