On the fundamental group of Rauzy fractals

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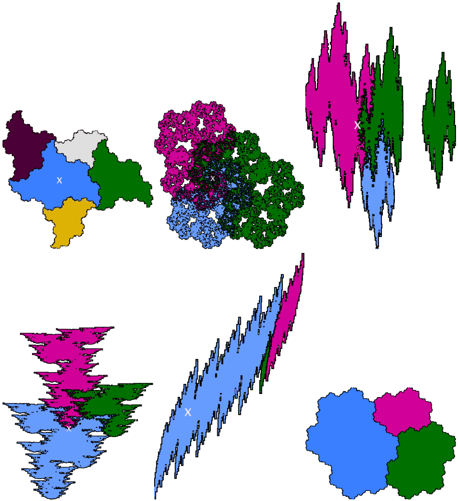
Many shapes for Rauzy fractals...

Several properties on examples [Rauzy, Akiyama, Canterini, Messaoudi, Feng-Furukado-Ito-Wu, Sirvent, Thus.]

0 inner point? Hausdorff dimension of the boundary?

Connectivity? Homeomorphic to a disk? Fundamental group?



Definitions

- Substitution. σ endomorphism of the free monoid $\{0, \dots, n\}^*$.
  $$\sigma : \quad 1 \to 12 \quad 2 \to 13 \quad 3 \to 1. \qquad (\beta^3 = \beta^2 + \beta + 1)$$
- Primitivity The abelianized matrix $\mathbf{M}$ is primitive.
- Periodic points If σ is primitive, there exists a periodic point w
  $$\sigma^\nu(w) = w.$$
- Pisot unit hypothesis The dominant eigenvalue β of $\mathbf{M}$ is a unit Pisot number.
  $$\sigma : \quad 1 \to 12 \quad 2 \to 3 \quad 3 \to 1 \quad 4 \to 5 \quad 5 \to 1 \qquad (\beta^3 = \beta + 1)$$
- (Ir)reducibility We denote by $d \leq n$ the algebraic degree of β and Min_β its minimal polynomial.
  If $d \neq n$, the substitution is said to be reducible.
- Decomposition of $\mathbb{R}^n$:
  Expanding line $\mathbb{H}_e$.
  β-contracting space $\mathbb{H}_c$ (generated by the eigenvectors of β Galois conjugates).
  Supplementary space $\mathbb{H}_o$ (generated by other eigenvectors)
- Beta-projection h: projection onto the contracting space, parallel to the other spaces.

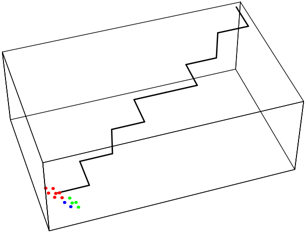
The beta-projection h retains the part of a vector lying on eigendirections for contracting conjugates of β

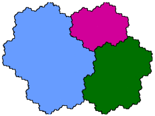
$$\forall w \in \mathcal{A}^*, \ \pi(\mathbf{l}(\sigma(w))) = \mathbf{h}\pi(\mathbf{l}(w)).$$

Projecting $\circ$ Abelianizing $\circ$ Substituting $\iff$ Projecting $\circ$ Contracting by $\mathbf{h}$ $\circ$ Abelianizing

Rauzy Fractal / Central Tile

- Compute a periodic point. $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 1$
  1213121121312121312113131213121121312121121312112131212112113...
- Draw a stair.
- Project it on the contracting space $\mathbb{H}_c$.
- Closure





Definition

$$\mathcal{T}_\sigma = \overline{\{\pi(\mathbf{l}(u_0 \cdots u_{i-1}));\ i \in \mathbb{N}\}}.$$

Subtile: $\mathcal{T}(a) = \overline{\{\pi\left(\mathbf{l}(u_0 \cdots u_{i-1})\right);\ i \in \mathbb{N},\ u_i = a\}}.$

Topology

- $\mathcal{T}$ is compact in $\mathbb{H}_c$. $\mathbb{H}_c$ is a $(d-1)$-Euclidean space, where d is the algebraic degree of β.
  Its interior is non empty. It has a non-zero measure in $\mathbb{H}_c$ [Sirvent-Wang].
- Each subtile is the closure of its interior [Sirvent-Wang].
- Subtiles are measurably disjoint if the substitution satisfies the strong coincidence condition [Arnoux-Ito].

Self-similarity[Arnoux-Ito]

The subtiles of $\mathcal{T}$ satisfy a Graph Iterated Function System:

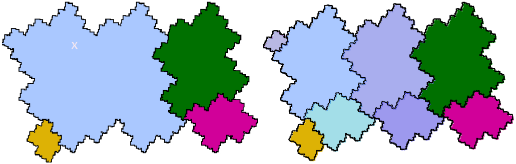
$$\mathcal{T}(a) = \bigcup_{b\in\mathcal{A},\ \sigma(b)=pas} \mathbf{h}(\mathcal{T}(b)) + \pi(\mathbf{l}(p))$$

$$\mathcal{T}(1) = \mathbf{h}[\mathcal{T}(1) \cup (\mathcal{T}(1) + \pi\mathbf{l}(e_1)) \cup \mathcal{T}(2) \cup (\mathcal{T}(2) + \pi\mathbf{l}(e_1)) \cup \mathcal{T}(4)],$$

$$\mathcal{T}(2) = \mathbf{h}(\mathcal{T}(1) + 2\pi\mathbf{l}(e_1)),$$

$$\mathcal{T}(3) = \mathbf{h}(\mathcal{T}(2) + 2\pi\mathbf{l}(e_1)),$$

$$\mathcal{T}(4) = \mathbf{h}(\mathcal{T}(3))$$



$\sigma(1) = 112,\ \sigma(2) = 113,\ \sigma(3) = 4,\ \sigma(4) = 1$

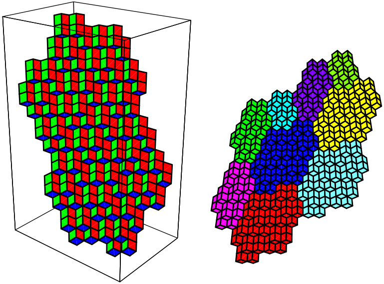
(Rauzy, Arnoux-Ito, Akiyama, Sirvent-Wang, Canterini-Siegel, Berthé-Siegel)

Covering the contracting space

- We consider the projection of points of $\mathbb{Z}^n$ that are nearby the contracting space along the expanding β-direction.

$$\Gamma_{srs} = \{[\pi(\mathbf{x}), i] \in \pi(\mathbb{Z}^n) \times \mathcal{A},\ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}.$$

  The distance of $\mathbf{x}$ to the contracting space along the expanding direction is smaller than the lenghth of the projection of the i-th canonical vector on the expanding direction.
- This set is self-similar, aperiodic and locally finite.
- For each pair $[\pi(\mathbf{x}), i]$ we draw a copy of $\mathcal{T}(i)$ in $\pi(\mathbf{x})$.



Covering [Ito-Rao,Barge-Kwapisz]

The set of tiles $\mathcal{T}(i) + \gamma$ with $(\gamma, i) \in \Gamma_{srs}$ covers the contracting space $\mathbb{H}_c$ with a constant cover degree.

Boundary graph

- Consider the intersection of two tiles in the covering $\mathcal{I} = \mathcal{T}(a) \cap (\pi(\mathbf{x}) + \mathcal{T}(b))$.
- Decompose each tile and re-order the intersection

$$\begin{aligned}\mathcal{I} &= \bigcup_{\sigma(a_1)=p_1 a s_1} \mathbf{h}[\mathcal{T}(a_1) + \pi\mathbf{l}(p_1)] \bigcap \bigcup_{\sigma(b_1)=p_2 b s_2} \mathbf{h}[\mathcal{T}(b_1) + \pi\mathbf{l}(p_2)] + \pi(\mathbf{x}). \\ &= \bigcup \mathbf{h}\pi\mathbf{l}(p_1) + \mathbf{h}\left[\mathcal{T}(a_1) \cap (\mathcal{T}(b_1) + \underbrace{\pi\mathbf{l}(p_2) - \pi\mathbf{l}(p_1) + \mathbf{h}^{-1}\pi(\mathbf{x})}_{=\pi(\mathbf{x}_1)})\right]\end{aligned}$$

Graph

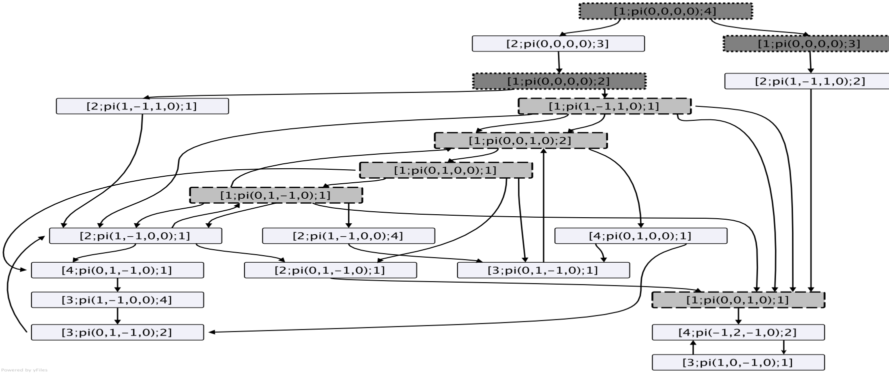
The nodes are denoted by $(a, \pi(\mathbf{x}), b)$ and correspond to intersections $\mathcal{T}(a) \cap (\pi(\mathbf{x}) + \mathcal{T}(b))$.
There is an edge between two nodes if the target intersection appears in the decomposition of the origin intersection.

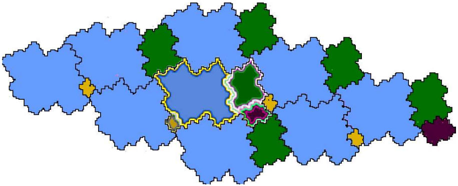
$$(a, \pi(\mathbf{x}), b) \to (a_1, \pi(\mathbf{x}_1), b_1)$$

- The graph is finite.
- The intersection $\mathcal{T}(a) \cap (\pi(\mathbf{x}) + \mathcal{T}(b))$ is non-empty iff the graph contains an infinite walk issued from $(a, \pi(\mathbf{x}), b)$.

Proof (a) If $\mathcal{I}$ is non-empty, at least one of the target is non-empty.
(b) There are only a finite number of non-empty intersections since the covering has a finite degree and$\mathcal{T}$ is bounded.

Example

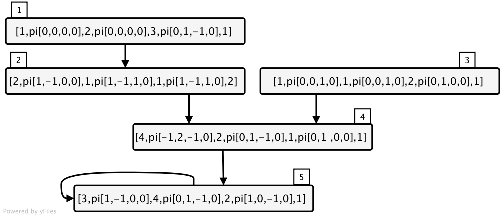


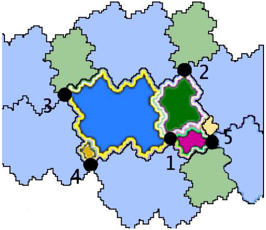


There are 8 nodes with the shape $[1, \pi(\mathbf{x}), b]$: hence $\mathcal{T}(1)$ has 8 neighbors $\pi(\mathbf{x}) + \mathcal{T}(b)$ in the covering.

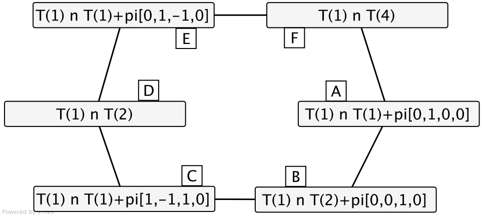
Derived graphs

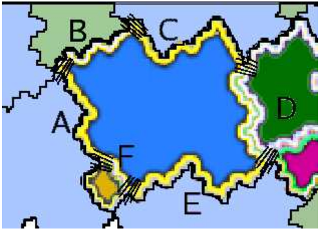
- Triple points graph We consider intersections between three tiles in the covering.
- Quadruple points graph Intersections between four tiles: only 5 quadruples points in the example .





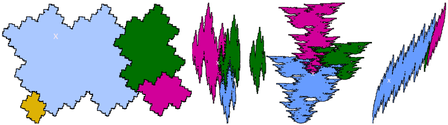
- The connectivity graph describe adjacencies of pieces of the boundary of a subtile $\mathcal{T}(i)$.





Applications

- Checking tiling and Box/Haussdorf dimension of the boundary: compute the dominant eigenvalue of the boundary graph (for $\sigma(1) = 112, \sigma(2) = 123,\ \sigma(3) = 4,\ \sigma(4) = 1$. the dimension is 1.1965).
- Connectivity ($d = 3$): stated in terms of connectivity graphs (non-connected for $\sigma(1) = 3, \sigma(2) = 23,\ \sigma(3) = 31223$)
- 0 inner point: related to a zero-surrounding graph (0 is not an inner point for $\sigma(1) = 123, \sigma(2) = 1,\ \sigma(3) = 31$)
- Homeomorphic to a disc ($d = 3$)
  (yes for $\sigma(1) = 112, \sigma(2) = 123,\ \sigma(3) = 4,\ \sigma(4) = 1$,
  no for $\sigma(1) = 1112, \sigma(2) = 1113,\ \sigma(3) = 1$.)
  - All connectivity graphs are loops
  - All connectivity graphs of the decomposition of tiles are lines.
  - Three-tiles intersections are single points.

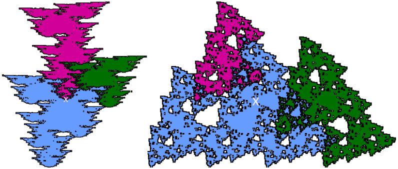


Non trivial fundamental group?

Theorem

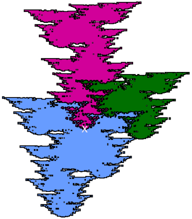
Assume that $d = 3$. The fundamental group of each $\mathcal{T}(i)$ is non-trivial as soon as

- The tiling property is satisfied;
- All $\mathcal{T}(i)$'s are connected;
- There are a finite number of quadruple points;
- There exists a triple point node $[i, i_1, \gamma_1, \mathcal{T}(i_2) + \gamma_2]$ leading away an infinity of walks.
- There exists three translations vectors such that the patterns $([\mathbf{v}, i], [\gamma_1 + \mathbf{v}, i_1], [\gamma_2 + \mathbf{v}, i_2])$, $([\mathbf{v}', i], [\gamma_1 + \mathbf{v}', i_1], [\gamma_2 + \mathbf{v}', i_2])$ and $([\mathbf{v}'', i], [\gamma_1 + \mathbf{v}'', i_1], [\gamma_2 + \mathbf{v}'', i_2])$ lie at the boundary of a finite inflation of $\mathbf{E}_1(\sigma)$.

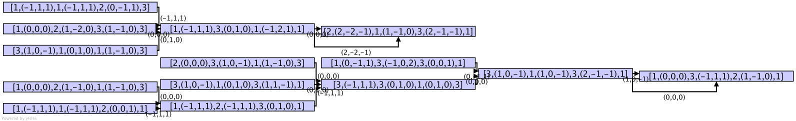


With additional properties, the fundamental group is not free and uncountable.

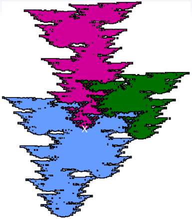
Example



- Finite number of quadruple points;
- A node $[i, i_1, \gamma_1, \mathcal{T}(i_2) + \gamma_2]$ in the triple points graph issues in an infinite number of walks.



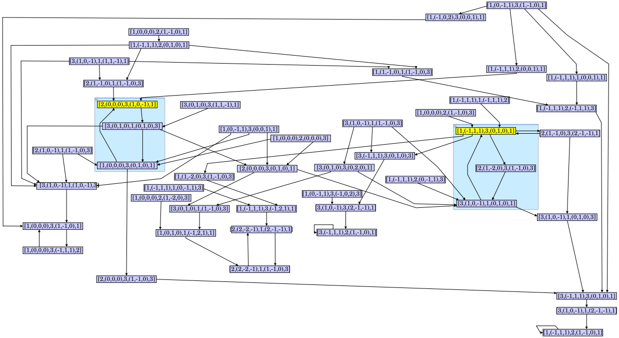
Example



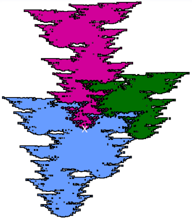
- Finite number of quadruple points;
- A node $[i, i_1, \gamma_1, \mathcal{T}(i_2) + \gamma_2]$ in the triple points graph issues in an infinite number of walks.

Consider the node $[2, \mathbf{0}, 3, \pi(1, 0, -1), 1]$.

It corresponds to the intersection $\mathcal{T}(2) \cap \mathcal{T}(3) \cap (\pi(1, 0, -1) + \mathcal{T}(1))$



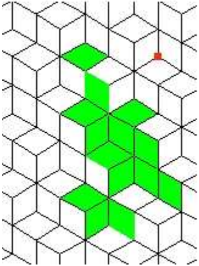
Example



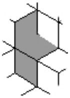
- Finite number of quadruple points;
- A node $[i, i_1, \gamma_1, \mathcal{T}(i_2) + \gamma_2]$ in the triple points graph issues in an infinite number of walks.

- There exists three translations vectors such that the three patterns $([\mathbf{v}, i], [\gamma_1 + \mathbf{v}, i_1], [\gamma_2 + \mathbf{v}, i_2])$, $([\mathbf{v}', i], [\gamma_1 + \mathbf{v}', i_1], [\gamma_2 + \mathbf{v}', i_2])$ et $([\mathbf{v}'', i], [\gamma_1 + \mathbf{v}'', i_1], [\gamma_2 + \mathbf{v}'', i_2])$ lie at the boundary of a finite inflation $\mathbf{E}_1(\sigma)^K[\mathbf{0}, i]$.

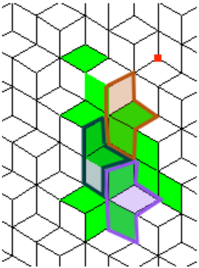
Consider the node $[2, \mathbf{0}, 3, \pi(1, 0, -1), 1]$.
It corresponds to the intersection $\mathcal{T}(2) \cap \mathcal{T}(3) \cap (\pi(1, 0, -1) + \mathcal{T}(1))$



$\mathbf{E}_1(\sigma)^4[\mathbf{0}, 2]$



Pattern
$[\mathbf{0}, 2][\mathbf{0}, 3]$
$[\pi(1, 0, -1), 1]$



Proof of the theorem I

Lemma [Luo, Thus.]

Let $B_0, B_1, B_2 \subset \mathbb{R}^2$ be locally connected continuum such that

(i) Interiors are disjoints $\mathrm{int}(B_i) \cap \mathrm{int}(B_j) = \emptyset$, $i \neq j$.

(ii) Each B_i is the closure of its interior $(0 \leq i \leq 2)$.

(iii) $\mathbb{R}^2 \setminus \mathrm{int}(B_i)$ is locally connected $(0 \leq i \leq 2)$.

(iv) There exist $x_1, x_2 \in B_0 \cap B_1 \cap B_2$ with $x_1 \in \mathrm{int}(B_0 \cup B_1 \cup B_2)$.

Then there exists $i \in \{0, 1, 2\}$ such that $B_i \cup B_{i+1}$ has a bounded connected component U with $U \cap \mathrm{int}(B_{i+2}) \neq \emptyset$.

In other words: if we consider three "suitable" sets that intersect simultaneously at least twice and one triple point is in the interior of the union, then a part of one set is surrounded by the two others.

- Apply this lemma to the three-tiles intersections given by the assumption. The infinite number of walks and the finite number of quadruple points ensures that a triple point is an inner point of the union.
- We use the last condition to ensure that the part of the third tile inside the hole is actually outside $\mathcal{T}(i)$.

Proof of the theorem II

- Since an infinite number of walks is issued from $[i, i_1, \gamma_1, \mathcal{T}(i_2) + \gamma_2]$ and quadruple points are finite, the intersection $\mathcal{T}(i) \cap (\mathcal{T}(i_1) + \gamma_1) \cap (\mathcal{T}(i_2) + \gamma_2)$ contains at least two distinct points and one lies in the interior of $\mathcal{T}(i) \cup (\mathcal{T}(i_1) + \gamma_1) \cup (\mathcal{T}(i_2) + \gamma_2)$.
- The lemma can be used. There exists a bounded connected component U_0 such that:
$$\mathbf{z} \in \mathrm{int}(\mathcal{T}(i_\ell) + \gamma_\ell) \cap \mathcal{U}_0 \qquad \mathcal{U}_0 \subset \mathcal{C}((\mathcal{T}(i_{\ell+1}) + \gamma_{\ell+1}) \cup (\mathcal{T}(i_{\ell+2}) + \gamma_{\ell+2})$$
- $\mathcal{T}(i_\ell) + \gamma_\ell + \mathbf{v}_\ell$ does not appear in the decomposition of $\mathbf{h}^{-N}\mathcal{T}(i)$ since
$$\mathbf{h}^{-N}\mathcal{T}(i) = \bigcup_{[\gamma,k] \in \mathbf{E}_1^N[0,i]} (\mathcal{T}(k) + \gamma).$$
By the tiling property, $\mathrm{int}(\mathcal{T}(i_\ell) + \gamma_\ell + \mathbf{v}_\ell)$ is disjoint from $\mathbf{h}^{-N}\mathcal{T}(i)$.
$$\mathbf{z} + \mathbf{v}_\ell \notin \mathbf{h}^{-N}\mathcal{T}(i). \implies \mathbf{z} + \mathbf{v}_\ell \in U \subset \mathcal{C}(\mathbf{h}^{-N}\mathcal{T}(i))$$
where U is a bounded connected component.
$$(\mathcal{T}(i_{\ell+1}) + \gamma_{\ell+1} + \mathbf{v}_\ell) \cup (\mathcal{T}(i_{\ell+2}) + \gamma_{\ell+2} + \mathbf{v}_\ell) \subset \mathbf{h}^{-N}\mathcal{T}(i) \implies \mathcal{U} \subset \mathcal{U}_0$$
- Therefore $\mathcal{U}$ is bounded.
- Then the complementary set to $\mathcal{T}(i)$ has at least a bounded connected component.

To be continued

- Changing the dimension: no more Jordan theorem available?
- Which properties are invariant by invertible substitution?
- Application to beta-numeration or diophantine approximation?