Balance properties of infinite words associated with quadratic Pisot numbers

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Plan of the talk

Balance properties of infinite words associated with quadratic Pisot numbers

1. Simple and non-simple quadratic Pisot numbers
2. The set of β-integers, infinite word associated to a quadratic Pisot number
3. Balance properties: k-balanced word
4. Theorem
5. Sketch of the proof

Quadratic Pisot numbers

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- *finite*, $d_\beta(1) = pq$, $p \geq q \geq 1$
- *eventually periodic*, $d_\beta(1) = pq^\omega$, $p > q \geq 1$ ($= pq^\omega$ denotes $pqqq \cdots$)

(no other possibility)

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- *eventually periodic*, $d_\beta(1) = pq^\omega$, $p > q \geq 1$
  - β is called a **non-simple** quadratic Pisot number.

β-integers

A β-**expansion** of $x \geq 0$ is a representation of the form

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots ,$$

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We denote: $\langle x \rangle_\beta = x_k x_{k-1} \cdots x_0 \bullet x_{-1} x_{-2} \cdots$

The set of non-negative β-**integers**:

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Theorem:
There are exactly two types of distances between neighboring points of $\mathbb{Z}_\beta^+$ on the real line, namely

- $\Delta_A = 1$,
- $\Delta_B = \beta - \lfloor\beta\rfloor$.

Generating substitutions for the word u_β

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- If β is *simple*, then u_β is a fixed point of

$$\begin{aligned} \varphi(A) &= A^p B, \\ \varphi(B) &= A^q, \quad p \geq q \geq 1. \end{aligned}$$

$$A \mapsto A^p B \mapsto (A^p B)^p A^q \mapsto \cdots$$

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- If β is *non-simple*, then u_β is a fixed point of

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Notation:

- *length* of a word $w = w_1 w_2 \cdots w_n$: $|w| = n$
- the number of letters A in the word w: $|w|_A$
- *factor* of the word v (finite or infinite): $v = w^{(1)} w w^{(2)}$
  - *prefix* of v: $v = w w^{(2)}$
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Definition: A word u in the binary alphabet $\mathcal{A} = \{A, B\}$ is k-balanced, if for every pair of factors w, $\hat{w}$ of u, it holds

$$|w| = |\hat{w}| \quad \Rightarrow \quad \big|\, |w|_A - |\hat{w}|_A \,\big| \leq k\,.$$

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Questions:

1. Are the words u_β k-balanced for some k?
2. If yes, what is the **minimal** k?

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For every quadratic Pisot number β there **is** a $k \in \mathbb{N}$ such that u_β is k-balanced.

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Theorem (our result)

- *If β is a quadratic simple Pisot number, $d_\beta(1) = pq$, then*

$$u_\beta \ \text{is} \ \left(1 + \left\lfloor \frac{p-1}{p+1-q} \right\rfloor\right)\text{-balanced}.$$

Balance properties of u_β

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- *If β is a quadratic non-simple Pisot number, $d_\beta(1) = pq^\omega$, then*

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- *These bounds are optimal, i.e. they cannot be improved.*

We define sequences $\left\{u_\beta^{(n)}\right\}_{n=1}^{\infty}$ and $\left\{w_\beta^{(n)}\right\}_{n=1}^{\infty}$ of factors of u_β:

$$\begin{aligned} w_\beta^{(1)} &= B \\ w_\beta^{(n)} &= B\varphi(w_\beta^{(n-1)}) \quad \text{for } n \in \mathbb{Z},\ n \geq 2, \end{aligned}$$

$$u_\beta^{(n)} = \text{prefix of } u_\beta \text{ of the length } \left|w_\beta^{(n)}\right|, \quad n \in \mathbb{N}$$

Proof of the Theorem (non-simple Pisot case) - part 1 of 3

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Main idea:

- $\left|u_\beta^{(n)}\right| = \left|w_\beta^{(n)}\right|$
- $u_\beta^{(n)}$ contains "many" letters A, $w_\beta^{(n)}$ contains "many" letters B.

The important property of the pairs $u_\beta^{(n)}, w_\beta^{(n)}$:

Lemma

The difference $|u_\beta^{(n)}|_A - |w_\beta^{(n)}|_A$ *is maximal in the following sense: If* v, v' *is a pair of factors of* u_β *of the same length and*

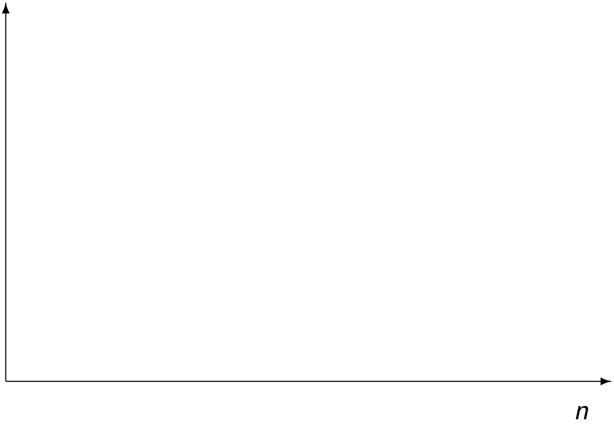
$$|v|_A - |v'|_A \quad > \quad |u_\beta^{(n)}|_A - |w_\beta^{(n)}|_A\,,$$

then

$$|v| = |v'| \quad > \quad |u_\beta^{(n)}| = |w_\beta^{(n)}|$$

Proof of the Theorem - part 3 of 3

Behavior of the difference $|u_\beta^{(n)}|_A - |w_\beta^{(n)}|_A$



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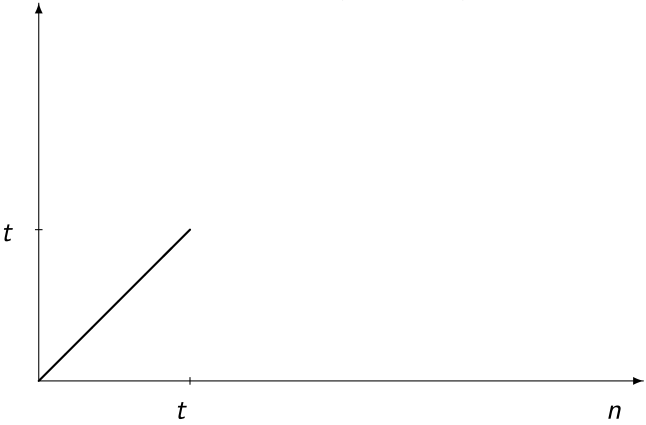

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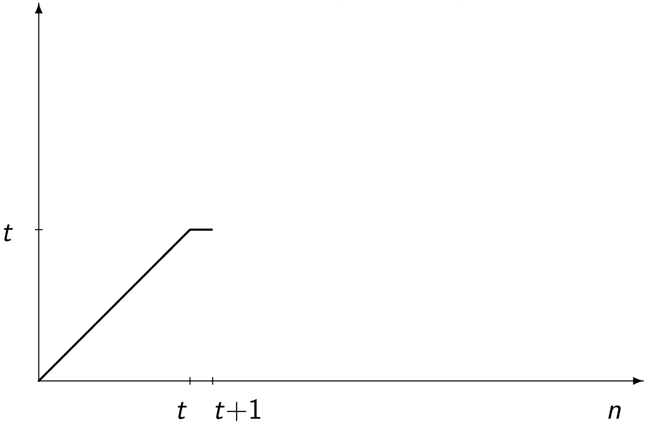

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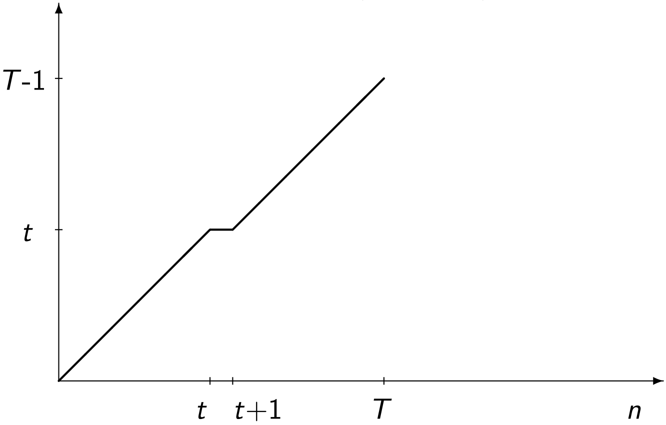

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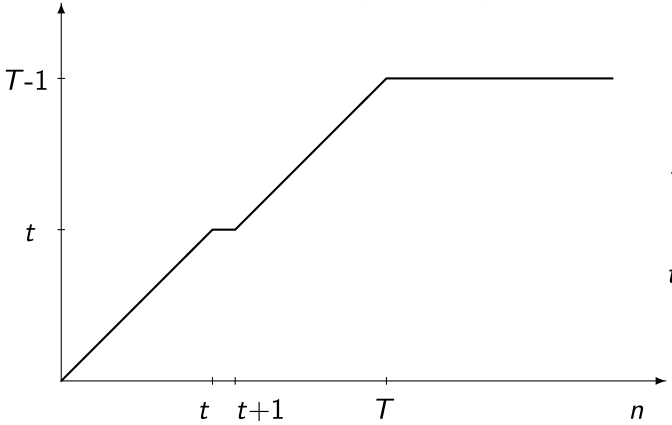

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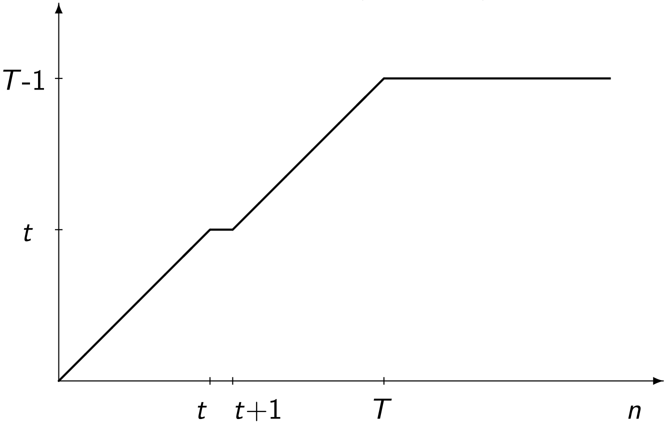

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$$\Rightarrow \quad k = T - 1 = \left\lceil \frac{p-1}{q} \right\rceil$$

Final remarks

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- The idea seems to be generalizable for certain Pisot numbers of higher degrees.