Rauzy fractals and Equi-distribution of Galois- and beta-conjugates of Parry Numbers near the unit circle

Jean-Louis Verger-Gaugry

Prague, Journées Numération,
Doppler Institute for Mathematical Physics and Applied Mathematics

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Contents

Contents

Let $k \geq 2$. $\beta (= \beta_k)$ is the dominant root of the minimal polynomial

$$P_\beta(X) = X^3 - (k+2)X^2 + 2kX - k.$$

We have : $k < \beta_k < k+1$ and $\lim_{k \to +\infty} (\beta_k - k) = 0$. The length of $d_{\beta_k}(1)$ is $2k+2 = d_P$;

$$f_{\beta_k}(z) = -1 + kz + \sum_{i=2}^{k-1} \left((i-1)z^i + (k-i+1)z^{k+i+1} \right) + kz^k + z^{k+1} + kz^{2k+2}$$

is minus the reciprocal polynomial of the Parry polynomial $n^*_\beta(X)$.
$k = 30$: the beta-conjugates are the roots of $(\phi_2(X)\phi_3(X)\phi_6(X)\phi_{10}(X)\phi_{30}(X)\phi_{31}(X)) \times (\phi_{10}(-X)\phi_{30}(-X))$.

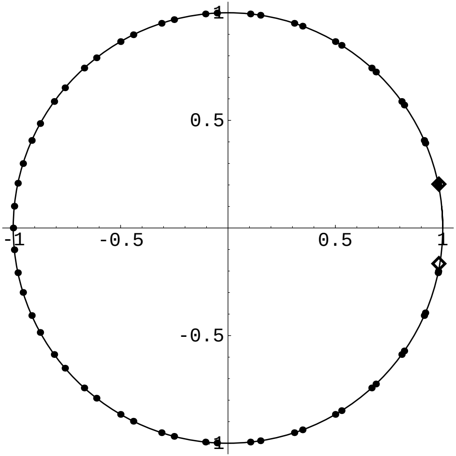

FIG.: Galois conjugates ($\diamond$) and beta-conjugates ($\bullet$) of the cubic Pisot number $\beta = 30.0356\ldots$, dominant root of $X^3 - 32X^2 + 60X - 30$.

Contents

$\beta > 1$ Perron number if algebraic integer and all its Galois conjugates $\beta^{(i)}$ satisfy : $|\beta^{(i)}| < \beta$ for all $i = 1, 2, \ldots, d-1$ (degree $d \geq 1$, with $\beta^{(0)} = \beta$).
Let $\beta > 1$. Rényi β-expansion of 1

$$d_\beta(1) = 0.t_1 t_2 t_3 \ldots \qquad \text{and corresponds to} \qquad 1 = \sum_{i=1}^{+\infty} t_i \beta^{-i} ,$$

$t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta\{\beta\} \rfloor = \lfloor \beta T_\beta(1) \rfloor, t_3 = \lfloor \beta\{\beta\{\beta\}\} \rfloor = \lfloor \beta T_\beta^2(1) \rfloor, \ldots$ The digits t_i belong to $\mathcal{A}_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$.

Parry number : if $d_\beta(1)$ is finite or ultimately periodic (i.e. eventually periodic) ; in particular, simple if $d_\beta(1)$ is finite.

Lothaire : a Parry number is a Perron number.

Dichotomy : set of Perron numbers

$$\mathbb{P} \; = \; \mathbb{P}_P \cup \mathbb{P}_a$$

Exploration of this dichotomy by the Erdős-Turán approach and its improvements (Mignotte, Amoroso) applied to

$$f_\beta(z) := \sum_{i=0}^{+\infty} t_i z^i \qquad \text{for } \beta \in \mathbb{P}, z \in \mathbb{C},$$

with $t_0 = -1$, where $d_\beta(1) = 0.t_1 t_2 t_3 \ldots$, for which $f_\beta(z)$ is a rational fraction if and only if $\beta \in \mathbb{P}_P$.

Beta-conjugates : D. Boyd 1996

Theorem (Szegő)

A Taylor series $\sum_{n \geq 0} a_n z^n$ with coefficients in a finite subset S of $\mathbb{C}$ is either equal to

(i) *a rational fraction $U(z) + z^{m+1} \dfrac{V(z)}{1 - z^{p+1}}$ where $U(z) = -1 + \sum_{i=1}^{m} b_i z^i$, $V(z) = \sum_{i=0}^{p} e_i z^i$ are polynomials with coefficients in S and $m \geq 1, p \geq 0$ integers, or*

(ii) *it is an analytic function defined on the open unit disk which is not continued beyond the unit circle (which is its natural boundary).*

Dichotomy of Perron numbers β <—> dichotomy of analytical functions $f_\beta(z)$.

$$\mathcal{B} := \{f(z) = 1 + \sum_{j=1}^{\infty} a_j z^j \mid 0 \leq a_j \leq 1\}$$

functions analytic in the open unit disk $D(0,1)$.

$$\mathcal{G} := \{\xi \in D(0,1) \mid f(\xi) = 0 \text{ for some } f \in \mathcal{B}\}$$

and

$$\mathcal{G}^{-1} := \{\xi^{-1} \mid \xi \in \mathcal{G}\}.$$

External boundary $\partial\mathcal{G}^{-1}$ of $\mathcal{G}^{-1}$: curve with a cusp at $z = 1$, a spike on the negative real axis, $= \left[-\frac{1+\sqrt{5}}{2}, -1\right]$, and is fractal at an infinite number of points.

$$\Omega := \mathcal{G}^{-1} \cup \overline{D(0,1)}.$$

Theorem (Solomyak)

The Galois conjugates ($\neq \beta$) and the beta-conjugates of all Parry numbers β belong to Ω, occupy it densely, and

$$\mathbb{P}_P \cap \Omega = \emptyset.$$

$$f_\beta(z) = -1+\sum_{i=1}^{\infty} t_i z^i = (-1+\beta z)\Big(1+\sum_{j=1}^{\infty} T_\beta^j(1) z^j\Big), \qquad |z| < 1,$$

-> the zeros $\neq \beta^{-1}$ of $f_\beta(z)$ are those of $1 + \sum_{j=1}^{\infty} T_\beta^j(1) z^j$; but $1 + \sum_{j=1}^{\infty} T_\beta^j(1) z^j$ is a Taylor series which belongs to $\mathcal{B}$.

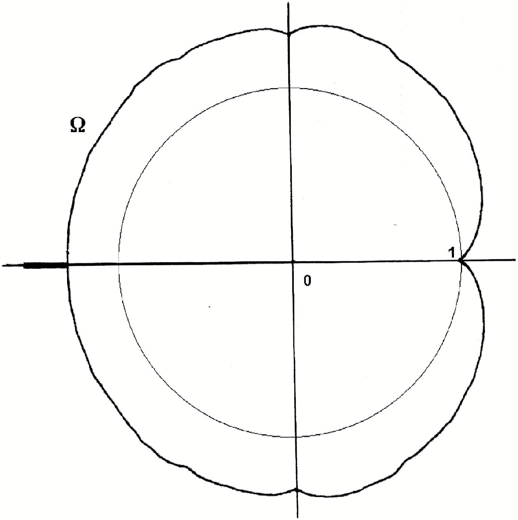

FIG.: Solomyak's set Ω.

-> phenomenon of high concentration and equi-distribution of Galois conjugates ($\neq \beta$) and beta-conjugates of a Parry number β occurs by clustering near the unit circle $\boxed{\text{in } \Omega}$.

Theorem

Let $\beta > 1$ be a Parry number. Let $\epsilon > 0$ and μ_ϵ the proportion of roots of the Parry polynomial $n_\beta^*(X)$ of β, with $d_P = \deg(n_\beta^*(X)) \geq 1$, which lie in Ω outside the annulus $\left(\overline{D(0,(1-\epsilon)^{-1})} \setminus D(0,(1-\epsilon))\right)$. Then

$$(i) \quad \mu_\epsilon \leq \frac{2}{\epsilon\, d_P}\left(\mathrm{Log}\|n_\beta^*\|_2 - \frac{1}{2}\mathrm{Log}\beta\right),$$

$$(ii) \quad \mu_\epsilon \leq \frac{2}{\epsilon\, d_P}\left(\mathrm{Log}\|n_\beta^*\|_1 - \frac{1}{2}\mathrm{Log}|n_\beta^*(0)|\right).$$

(i) Let $\mu_1 d_P$ the number of roots of $n_\beta^*(X)$ outside $\overline{D(0,(1-\epsilon)^{-1})}$ in Ω, except β (since $\beta \notin \Omega$). By Landau's inequality

$$M(f) \leq \|f\|_2 \qquad \text{for } f(x) \in \mathbb{C}[X]$$

applied to $n_\beta^*(X)$ we deduce

$$\beta(1-\epsilon)^{-\mu_1 d_P} \;\leq\; M(n_\beta^*) \;\leq\; \|n_\beta^*\|_2.$$

Since $-\mathrm{Log}(1-\epsilon) \geq \epsilon$,

$$\mu_1 \;\leq\; \frac{1}{\epsilon}\left(\frac{\mathrm{Log}\|n_\beta^*\|_2}{d_P} - \frac{\mathrm{Log}\beta}{d_P}\right).$$

Let $\mu_2 d_P$ the number of roots of $n^*_\beta(X)$ in $D(0, 1-\epsilon)$. Then

$$(1-\epsilon)^{-\mu_2 d_P} \leq M(n_\beta) \leq \|n_\beta\|_2 = \|n^*_\beta\|_2$$

by Landau's inequality applied to $n_\beta(X)$. We deduce

$$\mu_2 \ \leq \ \frac{1}{\epsilon} \, \frac{\mathrm{Log}\|n^*_\beta\|_2}{d_P}.$$

Since $\mu_\epsilon = \mu_1 + \mu_2$, we deduce the inequality.

(ii) Applying Jensen's formula,

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Log}\left|n_\beta^*(e^{i\phi})\right| d\phi - \mathrm{Log}\left|n_\beta^*(0)\right| = \sum_{|b_i|<1} \mathrm{Log}\frac{1}{|b_i|}$$

where (b_i) is the collection of zeros of $n_\beta^*(z)$. We have

$$\sum_{|b_i|<1} \mathrm{Log}\frac{1}{|b_i|} \geq \sum_{|b_i|<1-\epsilon} \mathrm{Log}\frac{1}{|b_i|} \geq \epsilon\, \mu_2 d_P .$$

Since $\max_{\phi\in[0,2\pi]} \left|n_\beta^*(e^{i\phi})\right| \leq \|n_\beta^*\|_1$,

$$\mu_2 \leq \frac{1}{\epsilon\, d_P}\left(\mathrm{Log}\|n_\beta^*\|_1 - \mathrm{Log}\left|n_\beta^*(0)\right|\right) .$$

The roots of $n_\beta(z)$ inside $D(0, 1-\epsilon)$ are the roots of $n^*_\beta(z)$ outside the closed disk $\overline{D(0,(1-\epsilon)^{-1})}$, including possibly β, so that their number is $\mu_1 d_P$ or $\mu_1 d_P + 1$.
Since $n^*_\beta(X)$ is monic, $|n_\beta(0)| = 1$. We apply Jensen's formula to $n_\beta(z)$

$$\mu_1 \leq \frac{1}{\epsilon\, d_P} \left(\mathrm{Log}\|n_\beta\|_1\right).$$

Since $\|n_\beta\|_1 = \|n^*_\beta\|_1$ and $\mu_\epsilon = \mu_1 + \mu_2$ -> claim.

Terminology "clustering near the unit circle" : if (β_i) is a sequence of Parry numbers, of Parry polynomials of respective degree $d_{P,i}$ which satisfies

$$\lim_{i \to +\infty} d_{P,i} = +\infty \quad \text{and} \quad \lim_{i \to +\infty} \frac{\mathrm{Log}\,\beta_i}{d_{P,i}} = 0,$$

then, since $\|n^*_{\beta_i}\|_2 \leq (d_{P,i}+1)^{1/2} \lceil \beta_i \rceil$, the proportion $\mu_{\epsilon,i}$ relative to β_i satisfies

$$\mu_{\epsilon,i} \leq \frac{1}{\epsilon} \left(\frac{\mathrm{Log}(d_{P,i}+1)}{d_{P,i}} + \frac{\mathrm{Log}\lceil \beta_i \rceil}{d_{P,i}} \right)$$

what shows, for $\epsilon > 0$, that

$$\mu_{\epsilon,i} \to 0, \qquad i \to +\infty.$$

The sufficient conditions for having convergence of $(\mu_{\epsilon,i})_i$ to zero do not imply that

- the corresponding sequence $(d_i)_i$ of the degrees of the minimal polynomials $P_{\beta_i}(X)$ tends to infinity ; on the contrary, this sequence may remain bounded, even stationary,
- the family of Parry numbers $(\beta_i)_i$ tends to infinity ; it may remain bounded or not

Define the radial operator $^{(r)} : \mathbb{Z}[X] \to \mathbb{R}[X]$,

$$R(X) = a_n \prod_{j=0}^{n} (X - b_j) \;\to\; R^{(r)}(X) = \prod_{j=0}^{n} \left(X - \frac{b_j}{|b_j|} \right).$$

-> roots on the unit circle.

This operator leaves invariant cyclotomic polynomials. It has the property : $P^{(r)} = (P^*)^{(r)}$ for all polynomials $P(X) \in \mathbb{Z}[X]$ and is multiplicative : $(P_1 P_2)^{(r)} = P_1^{(r)} P_2^{(r)}$ for $P_1(X), P_2(X) \in \mathbb{Z}[X]$.

Theorem (Mignotte)

Let (with $a_n \neq 0$, and $\rho_1, \rho_2, \ldots, \rho_n > 0$)

$$R(X) = a_n X^n + a_{n-1} X^{n-1} + \ldots + a_1 X + a_0 = a_n \prod_{j=1}^{n} (X - \rho_j e^{i\phi_j})$$

be a polynomial with complex coefficients, where $\phi_j \in [0, 2\pi)$ for $j = 1, \ldots, n$. For $0 \leq \alpha \leq \eta \leq 2\pi$, put $N(\alpha, \eta) = \mathrm{Card}\{j \mid \phi_j \in [\alpha, \eta]\}$. Let $k = \sum_0^\infty \frac{(-1)^{m-1}}{(2m+1)^2} = 0.916\ldots$ be Catalan's constant. Then

$$\left| \frac{1}{n} N(\alpha, \eta) - \frac{\eta - \alpha}{2\pi} \right|^2 \leq \frac{2\pi}{k} \times \frac{\tilde{h}(R)}{n}$$

where $\qquad \tilde{h}(R) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Log}^+ |R^{(r)}(e^{i\theta})| d\theta.$

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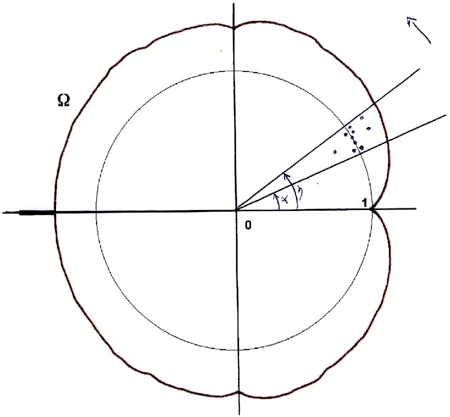

FIG.: Given an opening angle, a rotating sector contains the same number of roots of the Parry polynomial, up to Mignotte's discrepancy function. Angle is fixed by the geometry of Galois conjugates to detect beta-conjugates.

Denote $\text{dis}(R) = \frac{\tilde{h}(R)}{n}$. Call Mignotte's discrepancy function

$$C \cdot \text{dis}(R) = \frac{2\pi}{k} \times \frac{\tilde{h}(R)}{n}$$

with $C = \frac{2\pi}{k} = (2.619...)^2 = 6.859....$

-> dis(R) gives much smaller numerical estimates than Erdős-Turán's one : C= $16^2 = 256$ and $\text{dis}(R) = \frac{1}{n} \operatorname{Log} \frac{L(R)}{\sqrt{|a_0\, a_n|}}$.

Splitting :

$$\tilde{h}(n_\beta^*) = \tilde{h}(n_\beta) \le \tilde{h}(P_\beta) + \tilde{h}(\prod_{j=0}^{s} \Phi_{n_j}^{c_j}) + \tilde{h}(\prod_{j=0}^{q} \kappa_j^{\gamma_j}) + \tilde{h}(\prod_{j=0}^{u} g_j^{\delta_j}).$$

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Contents

Numerator of $f_\beta(z)$:

$$-n_\beta(z) = U(z), \qquad \text{resp.} \quad U(z)(1 - z^{p+1}) + z^{m+1}V(z).$$

Parry polynomial :

$$n^*_\beta(X) := P_\beta(X) \left(- \prod_{j=1}^{s} \left(\Phi_{n_j}(z)\right)^{c_j} \prod_{j=1}^{q} \left(\kappa_j(z)\right)^{\gamma_j} \prod_{j=1}^{u} \left(g_j(z)\right)^{\delta_j} \right)$$

where $P_\beta(X)$ = minimal polynomial of β.
where $\Phi_{n_j}(X) \in \mathbb{Z}[X]$ are irreducible and cyclotomic, with $n_1 < n_2 < \ldots < n_s$,
$\kappa_j(X) \in \mathbb{Z}[X]$ are irreducible and non-reciprocal,
$g_j(X) \in \mathbb{Z}[X]$ are irreducible, reciprocal and non-cyclotomic
-> Schinzel conjectures... Theorems.

$m \geq 0$, non-simple :

$$n_\beta^*(X) = X^{m+p+1} - t_1 X^{m+p} - t_2 X^{m+p-1} - \ldots - t_{m+p} X - t_{m+p+1}$$
$$- X^m + t_1 X^{m-1} + t_2 X^{m-2} + \ldots + t_{m-1} X + t_m$$

Simple ($m \geq 1$) :

$$X^m - t_1 X^{m-1} - t_2 X^{m-2} - \ldots - t_{m-1} X - t_m$$

The Parry polynomial is of small height :

$$\lfloor \beta \rfloor \leq \mathrm{H}(n_\beta^*) \leq \lceil \beta \rceil$$

with all coefficients having a modulus $\leq \lfloor \beta \rfloor$ except possibly only one.
β simple :

$$\mathrm{H}(n_\beta^*) \; = \; \lfloor \beta \rfloor .$$

Theorem

Let β be a Parry number. If ξ is a beta-conjugate of β which is not a unit, then its multiplicity ν_ξ as root of the Parry polynomial $n_\beta^*(X)$ satisfies :

$$\nu_\xi \leq \frac{1}{\log 2}\Big(\log(\mathrm{H}(n_\beta^*)) - \log|N(\beta)|\Big).$$

Moreover, if

$$|N(\beta)| \geq \frac{\mathrm{H}(n_\beta^*)}{3},$$

then all beta-conjugates of β which are not units (if any) are simple roots of $n_\beta^*(X)$.

Corollary

The beta-conjugates of a Parry number $\beta \in (1,3)$ which are not units are always simple roots of the Parry polynomial of β.

$P_\beta(X)$ divides $n^*_\beta(X)$ and $\mathrm{H}(n^*_\beta) \in \{\lfloor\beta\rfloor, \lceil\beta\rceil\}$. Then

$$|\prod_{j=1}^{q} \left(\kappa_j(0)\right)^{\gamma_j}| \times |\prod_{j=1}^{u} \left(g_j(0)\right)^{\delta_j}| \;\leq\; \frac{\mathrm{H}(n^*_\beta)}{|N(\beta)|}.$$

If ξ is a beta-conjugate, not a unit, then,

$$|N(\xi)| \geq 2 \quad \text{implies} \quad 2^{\nu_\xi} \;\leq\; \frac{\mathrm{H}(n^*_\beta)}{|N(\beta)|}.$$

If $|N(\beta)| \geq \frac{\mathrm{H}(n_\beta^*)}{3}$ then

$$|\prod_{j=1}^{q} \left(\kappa_j(0)\right)^{\gamma_j}| \times |\prod_{j=1}^{u} \left(g_j(0)\right)^{\delta_j}| \leq 3,$$

which necessarily implies $\nu_\xi = 1$ for each beta-conjugate ξ of β which is not a unit.

$d_P = m + p + 1 =$ deg. of Parry polynomial $n_\beta^*(X)$

$1 + s + q + u =$ # of distinct factors

$1 + \sum_{j=1}^{s} c_j + \sum_{j=1}^{q} \gamma_j + \sum_{j=1}^{u} \delta_j =$ # of factors counted with multiplicities

$1 + \sum_{j=1}^{q} \gamma_j + \sum_{j=1}^{u} \delta_j =$ # of non-cyclotomic factors counted with multiplicities

$1 + q + u =$ # of its non-cyclotomic factors counted without multiplici

$\gamma + \sum_{j=1}^{q} \gamma_j =$ # of its non-reciprocal factors counted with multiplicites
$\gamma = 1$ if $P_\beta(X)$ is non-reciprocal,
$\gamma = 0$ if $P_\beta(X)$ is reciprocal

Gazeau+VG,

Theorem

Let $\beta > 1$ be a Pisot number of degree $d \geq 2$. Then

$$d_P \leq \# \left\{ x \in \mathbb{Z}^d \mid p_2(x) \in \frac{\mathrm{H}(n^*_\beta)}{\lfloor \beta \rfloor} \Omega', \;\; \pi_B(x) \cdot u_B \in \left[0, \frac{1}{\|B\|}\right) \right\}.$$

Better upper bound of d_P : the “box" Ω' replaced by the central tile (of the Rauzy fractal)
Topology of this central tile may be disconnected,... is a prominent ingredient for counting points of the lattice $\mathbb{Z}^d$ which are projected by p_2 to this central tile (P. Arnoux, A. Siegel, V. Berthé, G. Barat, S. Akiyama, J. Thuswaldner,...).

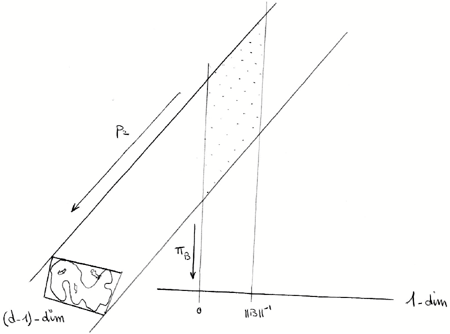

FIG.: Cut-and-project scheme in $\mathbb{R}^d$ over the set $\mathbb{Z}_\beta$ of β-integers. Slice of the band with lattice points over the central tile (Rauzy fractal).

The special sequence $(\Phi_{n_j})_{j=1,\ldots,s}$ of cyclotomic polynomials in the factorization of $n^*_\beta(X)$ is such that $\sum_{j=1}^{s} c_j \varphi(n_j) \leq d_P - d$, with $s \leq n_s$, where $\varphi(n)$ is the Euler function, and its determination is complemented by

Schinzel

Theorem

There exists a constant $C_0 > 0$ such that, for every Parry number β, the number s of distinct cyclotomic irreducible factors of the Parry polynomial of β satisfies

$$s \;\leq\; C_0 \sqrt{d_P}.$$

Amoroso : the assertion that the Riemann zeta function does not vanish for Rez $\geq \sigma + \epsilon$ is equivalent to the inequality

$$\tilde{h}\Big(\prod_{n=1}^{N} \Phi_n\Big) \ll N^{\sigma+\epsilon},$$

where $\sigma =$ supremum of the real parts of the non-trivial zeros of the Riemann zeta function, and $\sigma = 1/2$ if Riemann hypothesis (R.H.) true.
-> particular telescopic products of cyclotomic polynomials which appear in factorizations of Parry polynomials.

Amoroso

Theorem

Let $s \geq 1$. Let $c_1, \ldots, c_s$ integers ≥ 0 and $n_1 \leq n_2 \leq \ldots \leq n_s$ be a increasing sequence of positive integers. Assume R.H. true. Then there exists $A > 0$ such that

$$\mathrm{dis}\Big(\prod_{j=1}^{s} \Phi_{n_j}(X)^{c_j}\Big) \leq A \times \frac{\sqrt{n_s}}{\sum_{j=1}^{s} c_j \varphi(n_j)},$$

Let $N = n_s$. Let

$$G(X) = \prod_{n=1}^{N} \Phi_n(x)^{\sigma_n}$$

with

$$\sigma_n = \begin{cases} 0 & \text{if } n \notin \{n_1, n_2, \ldots, n_s\} \\ c_j & \text{if } n = n_j \text{ for } j \in \{1, 2, \ldots, s\} \end{cases}$$

for $n \geq 0$.

$$\tilde{h}(G) \leq \sqrt{\frac{\pi}{12}\sum_{m=1}^{N}\left(\sum_{j|m}\frac{\mu(j)}{j^2}\right)\left(\sum_{n\leq N/m}\sigma_{mn}\sum_{k|n}\frac{\mu(k)k}{n}\right)^2}$$

We have $0 \leq \sum_{j|m}\frac{\mu(j)}{j^2} \leq 1$ and, by Titchmarsh 14.25C,

$$\text{R.H. true} \iff \sum_{k\leq x}\mu(k) \ll x^{1/2+\epsilon}$$

for any ϵ.

Dobrowolski

Theorem

There exists a constant $C_1 > 0$ *such that for every Parry number* β *and* $\epsilon > 0$ *an arbitrary positive real number, then*

$$1 + \sum_{j=1}^{q} \gamma_j + \sum_{j=1}^{u} \delta_j \;\leq\; C_1 \left((d_P)^{\epsilon} (\log \|n^*_\beta\|_2^2)^{1-\epsilon} \right).$$

Cassels

Theorem

If χ is a beta-conjugate of a Parry number β such that the minimal polynomial $g(X)$ of χ is non-reciprocal, with $n = \deg(g)$, if $\chi_1, \ldots, \chi_{n-1}$ denote the Galois conjugates of $\chi = \chi_0$ (which are also beta-conjugates of β), then either

(i) $|\chi_j| > 1 + \frac{0.1}{n}$ *for at least one* $j \in \{0, 1, \ldots, n-1\}$, *or*

(ii) $g(X) = -g^*(X)$ *if* $|\chi_j| \leq 1 + \frac{0.1}{n}$ *holds for all* $j = 0, 1, \ldots, n-1$.

In the second case, since $g(X) = \prod_{j=0}^{n-1}(X - \chi_j) = -\prod_{j=0}^{n-1}(1 - \chi_j X)$ is monic, all the beta-conjugates χ_j of β ($j = 0, 1, \ldots, n-1$) are algebraic units, i.e. $|N(\chi_j)| = 1$.

Cassels

Theorem

If χ is a beta-conjugate of a Parry number β such that the minimal polynomial (of degree n) of χ is non-cyclotomic and where $\chi_1, \ldots, \chi_{n-1}$ denote the Galois conjugates of χ $(= \chi_0)$, if

$$|\chi_j| \leq 1 + \frac{0.1}{n^2} \qquad \text{for } j = 0, 1, \ldots, n-1,$$

then at least one of the beta-conjugates $\chi_0, \chi_1, \ldots, \chi_{n-1}$ of β has absolute value 1.

-> likely to be often applicable because of high concentration of beta-conjugates near the unit circle.

Smyth

Theorem

For every Parry number β, the inequality

$$\gamma + \sum_{j=1}^{q} \gamma_j < \frac{\log \|n_\beta^*\|_2}{\log \theta_0}$$

holds where $\theta_0 = 1.3247\ldots$ is the smallest Pisot number, dominant root of $X^3 - X - 1$, where $\gamma = 1$ if $P_\beta(X)$ is non-reciprocal and $\gamma = 0$ if $P_\beta(X)$ is reciprocal.

Corollary

If β is a Parry number for which the minimal polynomial is non-reciprocal and $d_\beta(1) = 0.t_1 t_2 t_3 \ldots$, of preperiod length $m \geq 0$ and period length $p+1$, satisfies (with $t_0 = -1$)

$$\left.\begin{array}{ll} \text{if } \beta \text{ is simple} & \sum_{j=0}^{m} t_j^2 \\ \text{if } \beta \text{ is non-simple} & \sum_{j=0}^{p} t_j^2 + (1+t_{p+1})^2 + \sum_{j=1}^{m} (t_j - t_{p+j+1})^2 \end{array}\right\} \leq \theta_0^4 =$$

then the Parry polynomial of β has no non-reciprocal irreducible factor in it ($\theta_0^4 = 3.0794...$).

Explicitely in the “simple" case : β for which $d_\beta(1)$ has necessarily the form

$$d_\beta(1) = 0.1\underbrace{00\ldots0}_{\delta}1$$

Algebraic integers $(\beta_\delta)_{\delta\geq 3}$ are Perron numbers studied by Selmer, roots of

$$X^{\delta+2} - X^{\delta+1} - 1.$$

The case $\delta = 0$ corresponds to the golden mean $\tau = (1+\sqrt{5})/2$.

Contents

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Absolute logarithmic height of a Parry number β :

$$h(\beta) := \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{v} [\mathbb{K}_v : \mathbb{Q}_v] \max(0, \mathrm{Log}|\beta|_v)$$

$\mathbb{K} :=$ algebraic number field generated by β, its Galois and beta-conjugates, so that $\mathbb{K} \supset \mathbb{Q}(\beta)$.
Weighted sum of Dirac measures :

$$\Delta_\beta := \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{\sigma:\mathbb{K}\to\mathbb{C}} \delta_{\{\sigma(\beta)\}}$$

where (images are Galois- or beta-conjugates) :

$$\sigma : \quad \beta \to \beta^{(i)} \qquad \text{or} \qquad \sigma : \quad \beta \to \xi_j .$$

Bilu

Theorem

Let $(\beta_i)_{i\geq 1}$ be a strict sequence of Parry numbers which satisfies

$$\lim_{i\to\infty} h(\beta_i) \to 0.$$

Then

$$\lim_{i\to\infty} \Delta_{\beta_i} = \nu_{\{|z|=1\}} \qquad \textit{Haar measure.}$$

Topology : a sequence of probability measures $\{\mu_k\}$ on a metric space S wealky converges to μ if for any bounded continuous function $f : S \to \mathbb{R}$ we have $(f, \mu_k) \to (f, \mu)$ as $k \to \infty$.

Strict : A sequence $\{\alpha_k\}$ of points in $\overline{\mathbb{Q}}^*$ is strict if any proper algebraic subgroup of $\overline{\mathbb{Q}}^*$ contains α_k for only finitely many values of k.

Bilu's ingredients : Erdős - Turán's Theorem, for sequences of Parry numbers which tend to 1.

Possible generalizations : to general convergent sequences of Parry numbers with

$$\lim_{i \to +\infty} d_{P,i} = +\infty \quad \text{and} \quad \lim_{i \to +\infty} \frac{\operatorname{Log} \beta_i}{d_{P,i}} = 0,$$

Need : p-adic control of the beta-conjugates to have convergence property for the measure : given by the forms of irreducible factors in the factorization of the Parry polynomials.

Rumely : reformulation in terms of Potential Theory, equilibrium measures, -> A. Granville Theorem. Like in electrostatics, repulsive effects between conjugates...

Contents

Idea : take advantage of this concentration and equi-distribution phenomenon to make Rauzy fractal constructions

using not only the Galois conjugates but also the beta-conjugates.

-> expectation : continuity theorems with β

Over adele space $\mathbb{A}_{\mathbb{K}_\beta}$, where $\mathbb{K}_\beta$ is the algebraic number field generated by the Galois- and the beta-conjugates of a Parry number β.

Classical Rauzy fractal : invariant under the action of some Galois group.